Reports of the Academy of Sciences of the USSR

1957

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1957. Vol. 115, No. 4

MATHEMATICS

V. N. GOL' DBERG

ON THE PERTURBATION OF LINEAR OPERATORS WITH PURELY DISCRETE SPECTRUM

(Presented by Academician S. L. Sobolev, 13 III 1957)

This note considers the question of the dependence of the eigenvalues and eigenvectors of an unbounded operator $H_\varepsilon = \varepsilon V + H_0$ ($\varepsilon \geqslant 0$), with domain of definition in a Hilbert space G, on the small parameter ε, where H_0 is a linear unbounded operator; V is a linear unbounded operator having a narrower domain of definition than H_0. Typical examples of such perturbations are the examples, known from physics, of differential operators containing a small parameter at the highest derivative. The general theorems proved by us are applied to certain differential operators.

Denote by D_0 the domain of definition of the operator H_0, and by D_1 the domain of definition of the operator V. It is assumed that $D_1 \subset D_0$ and D_1 is everywhere dense in G.

Let:

1) $(V\varphi, \varphi) \geqslant 0$ for $\varphi \in D_1$;
2) the operator H_ε ($\varepsilon \geqslant 0$) is self-adjoint and the operator H_0 is positive-definite;
3) the set of all φ from the domain of definition of the operator H_0 satisfying the condition $(H_0\varphi, \varphi) \leqslant 1$ is compact in G.

From assumptions 1)–3) it follows, by Rellich' s theorem ([1]), that the operator H_ε ($\varepsilon \geqslant 0$) has purely discrete spectrum. Denote by $\lambda_n(\varepsilon)$ the n-th eigenvalue and by $\varphi_n(\varepsilon)$ an arbitrary eigenvector of the operator H_ε corresponding to $\lambda_n(\varepsilon)$.

We shall assume the eigenvectors of the operator H_ε ($\varepsilon \geqslant 0$) to be orthonormal. Introduce the following notation:

$$J_0[\varphi] = (H_0\varphi, \varphi); \qquad D_0^{(1)} = \{\varphi \in D_0;\ \|\varphi\| = 1\};$$

$$D_1^{(1)} = \{\varphi \in D_1;\ \|\varphi\| = 1\};$$

$$D_1^{(i)} = \{\varphi \in D_1;\ \|\varphi\| = 1;\ (\varphi, \varphi_j(0)) = 0\} \quad (i > 1); \quad j = 1, 2, \dots, (i-1).$$

We shall say that, for some i, the approximation condition (a.c.) is satisfied if for every $\eta > 0$ there exists an element $\psi \in D_1^{(i)}$ such that $J_0[\psi] - \lambda_i(0) < \eta$. The a.c. is satisfied automatically when $D_1 = D_0$. The latter case is considered in Kato' s paper ([2]), some of whose results are analogous to the results of the following Theorem 1.

Theorem 1. Let the operator $H_\varepsilon = \varepsilon V + H_0$ satisfy conditions 1)–3), and let the a.c. be satisfied for $i = 1, 2, \dots$

Then:

1) $\lambda_i(\varepsilon) \to \lambda_i(0)$ as $\varepsilon \to 0$ $(i = 1, 2, \dots)$;
2) whatever sequence $\varepsilon_k \to 0$ as $k \to \infty$ and elements $\varphi_i(\varepsilon_k)$ may be, there exists a subsequence $\tilde{\varepsilon}_k \to 0$ as $k \to \infty$ and elements $\varphi_i(0)$ such that $J_0[\varphi_i(\tilde{\varepsilon}_k) - \varphi_i(0)] \to 0$ as $k \to \infty$ $(i = 1, 2, \dots)$.

The proof of Theorem 1 is carried out with the aid of the extremal properties of eigen-elements.

Corollary. Let the multiplicity of each eigenvalue of the operator H_0 be equal to one.

Then:

1) there exists an $\varepsilon_0(i)$ such that, for all $\varepsilon < \varepsilon_0(i)$, the multiplicity of the i-th eigenvalue of the operator H_ε is also equal to one.
2) $J_0[\varphi_i(\varepsilon) - \varphi_i(0)] \to 0$ as $\varepsilon \to 0$ $(i = 1, 2, \dots)$.

Theorem 2. Let the operator $H_\varepsilon = \varepsilon V + H_0$ satisfy conditions 1)–3). Suppose further that, for some natural number i, there exists a sequence $\varepsilon_k \to 0$ as $k \to \infty$ and elements $\varphi_j(\varepsilon_k), \varphi_j(0)$ $(j = 1, 2, \dots, i)$ such that:

1) $\lambda_j(\varepsilon_k) \to \lambda_j(0)$ as $k \to \infty$ $(j = 1, 2, \dots, i)$;
2) $\|\varphi_j(\varepsilon_k) - \varphi_j(0)\| \to 0$ as $k \to \infty$ $(j = 1, 2, \dots, i)$.

Then for $i = j$ the condition u. a. is fulfilled.

To verify the fulfillment of u. a., as will be seen from the examples given below, it is convenient to use Theorem 3.

Theorem 3. Let, for the operator $H_\varepsilon = \varepsilon V + H_0$, the following conditions be satisfied:

A. H_ε satisfies conditions 1)–3).

B. For every element $\varphi \in D_0^{(1)}$ there exists a sequence of elements $\psi_n \in D_1^{(1)}$ such that:

1) $\|\psi_n - \varphi\| \to 0$ as $n \to \infty$;
2) $J_0[\psi_n] \to J_0[\varphi]$ as $n \to \infty$.

Then for $i = 1, 2, ...$ the condition u. a. is fulfilled.

Example 1. Denote by H_ε $(\varepsilon > 0)$ the differential operator generated by the differential expression

$$l_\varepsilon[y] \equiv \varepsilon y^{(\mathrm{IV})} - \frac{d}{dx}\left[p(x)y'\right] + q(n)y$$

and the boundary conditions

$$y(a) = y'(a) = y(b) = y'(b) = 0.$$

Denote by H_0 the differential operator generated by the differential expression

$$l_0[y] \equiv -\frac{d}{dx}\left[p(x)y'\right] + q(x)y$$

and the boundary conditions

$$y(a) = y(b) = 0.$$

It is assumed that

$$p(x) \geq p_0 > 0; \qquad p(x) \in C^1_{[ab]}; \qquad q(x) \in C_{[ab]}.$$

To verify the fulfillment of u. a. we use Theorem 3. Since the multiplicity of each eigenvalue of the operator H_0 is equal to one, it follows from Theorem 1 that:

1) $\lambda_i(\varepsilon) \to \lambda_i(0)$ as $\varepsilon \to 0$ $(i = 1, 2, ...)$;
2) $y_i(x, \varepsilon) \to y_i(x, 0)$ as $\varepsilon \to 0$, uniformly with respect to $x \in [a, b]$ $(i = 1, 2, ...)$;
3) $$\int_a^b |y_i'(x, \varepsilon) - y_i'(x, 0)|^2 \, dx \to 0 \quad \text{as } \varepsilon \to 0 \quad (i = 1, 2, ...).$$

We note that assertions 1) and 2) are also contained in a note by V. B. Glazko ([3]).

Example 2. Let Ω be a domain of n-dimensional space, bounded by a sufficiently smooth surface S. Denote by H_0 the differential operator generated by the differential expression

$$l_0[u] \equiv -\Delta u \equiv -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

and the boundary condition $u|_S = 0$. Denote by H_ε $(\varepsilon > 0)$ the differential operator generated by the differential expression

$$l_\varepsilon[u] \equiv \varepsilon \Delta^2 u - \Delta u$$

and the boundary conditions

$$u\Big|_S = 0, \qquad \frac{\partial u}{\partial n}\Big|_S = 0.$$

The operator H_ε $(\varepsilon > 0)$ is positive-definite and can be extended to a self-adjoint one ([4], pp. 15–24). From the known theorems on the complete continuity of the embedding operator ([5]) it follows that condition 3 is satisfied.

The following theorem holds:

Theorem. Let the domain Ω be bounded by a smooth surface S. If the function $u(x)$ is twice continuously differentiable in $\overline{\Omega}$ and is equal to zero on S, then one can construct a function $v(x)$, twice continuously differentiable in $\overline{\Omega}$ and equal to zero in some boundary strip, so that the inequality

$$\int_\Omega \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right|^2 d\Omega < \varepsilon,$$

holds, where ε is any prescribed positive number ([4], p. 129).

To verify that u.a. is satisfied for $i = 1, 2, \ldots$, we use this theorem and Theorem 3.

Thus, Theorem 1 is valid.

The author expresses his gratitude to his scientific adviser A. G. Sigalov.

Gorky
State University

Received
12 III 1957

CITED LITERATURE

[1] S. G. Mikhlin, *Vestn. LGU*, No. 8, ser. mathemat., physic., chem., issue 3, 23 (1954).
[2] T. Kato, *Math. Ann.*, **125**, 5, 435 (1953).
[3] V. B. Glazko, *DAN*, **108**, No. 5 (1956).
[4] S. G. Mikhlin, *The Problem of the Minimum of a Quadratic Functional*, Moscow-Leningrad, 1952.
[5] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Leningrad, 1950, pp. 83–94.

Note: Figure translations are in progress. See original paper for figures.