THE PROBLEM OF THE PRESSURE OF A PUNCH ON AN ELASTIC HALF-PLANE WITH A CIRCULAR HOLE

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Fig. 1

Figure 1: Fig. 1

Abstract

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THEORY OF ELASTICITY

I. G. ARAMANOVICH

THE PROBLEM OF THE PRESSURE OF A PUNCH ON AN ELASTIC HALF-PLANE WITH A CIRCULAR HOLE

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1. In the solution of many important contact problems of the plane theory of elasticity it is usually assumed that one of the elastic bodies may be replaced by a half-plane. The works in this direction of L. A. Galin, N. I. Muskhelishvili, I. Ya. Shtaerman, and other authors are widely known.

In the present article we consider the contact problem of the pressure of a punch for the case in which the half-plane has a circular hole, symmetrically situated near the place of its contact with the punch. We shall assume that the elastic medium occupies the lower half-plane (Fig. 1); denote its boundary by L_0. Let on the segment $L_0(-a, a)$ a punch with a rectilinear base parallel to the axis Ox be rigidly attached to it, moving only vertically; the external forces acting on the punch are taken to be symmetric with respect to the axis Oy and reducible to the resultant

$$X = 0, \quad Y = -P.$$

Fig. 1

The elastic half-plane has a circular hole whose center is at the point $z = -ih$ $(h > 1)$, and whose radius is equal to unity. The contour of this hole (we denote it by L_1), as well as the part of the boundary L_0 not in contact with the punch (L_0''), are taken to be free from the action of external forces. The simply connected domains bounded by the contours L_j $(j = 0, 1)$ and containing the origin of coordinates will be denoted by S_j, and the doubly connected domain bounded by the contours L_0 and L_1 will be denoted by $S_{0,1}$. The direction of traversal of the contours L_j is indicated in Fig. 1.

Introduce the complex Kolosov-Muskhelishvili potentials (1) $\varphi_0(z)$ and $\psi_0(z)$, regular in the domain $S_{0,1}$; denote their derivatives by $\Phi_0(z)$ and $\Psi_0(z)$.

The boundary conditions of the problem posed may be written in the form

$$\varkappa\Phi_0(t) - \overline{\Phi_0(t)} - t\overline{\Phi_0'(t)} - \overline{\Psi_0(t)} = 0 \quad \text{on } L_0', \tag{1}$$

$$\Phi_0(t) + \overline{\Phi_0(t)} + t\overline{\Phi_0'(t)} + \overline{\Psi_0(t)} = 0 \quad \text{on } L_0'', \tag{2}$$

$$\Phi_0(t) + \left[\overline{\Phi_0(t)} - e^{-2i\varphi}\left(t\overline{\Phi_0'(t)} + \overline{\Psi_0(t)}\right)\right] = 0 \quad \text{on } L_1 \tag{3}$$

(φ is the angle between the normal to the circumference L_1 and the abscissa axis).

Making a slight modification of D. I. Sherman' s method (^{2,3}), denote the boundary values of the function $\Phi_0(t)$ on the contour L_1 by $\Omega(t)$, and introduce the function

$$\Phi(z) = \begin{cases} \Phi_0(z) + \dfrac{1}{2\pi i}\displaystyle\int_{L_1} \dfrac{\Omega(\tau)}{\tau - z}\, d\tau & (z \text{ in } S_{0,1}), \\ \dfrac{1}{2\pi i}\displaystyle\int_{L_1} \dfrac{\Omega(\tau)}{\tau - z}\, d\tau & (z \text{ in } S_1). \end{cases} \tag{4}$$

The function $\Phi(z)$, as follows from the Plemelj–Sokhotski formulas, is regular in the entire domain S_0. From (3) we then obtain that the function $\Psi(z)$, defined in the domain $S_{0,1}$ by the equality

$$\Psi(z) = \Psi_0(z) + \frac{1}{2\pi i}\int_{L_1} \frac{\overline{\Omega(\tau)}}{(\bar\tau + ih)^2(\tau - z)}\, d\tau - \frac{1}{2\pi i}\int_{L_1} \frac{\bar\tau\Omega(\tau)}{(\tau - z)^2}\, d\tau, \tag{5}$$

can also be continued regularly into the domain S_1.

2. Passing in (1) and (2) to $\Phi(t)$ and $\Psi(t)$, we obtain analogous equations for them, but with the addition, on the right-hand sides, of linear operators $F_1(t)$ and $F_2(t)$* of the auxiliary function $\Omega(t)$, which can be expressed through the quantities

$$\alpha_n = \frac{1}{2\pi i}\int_{L_1} \Omega(\tau)(\tau + ih)^n\, d\tau \quad (n = -\infty, \dots, \infty). \tag{6}$$

It follows from the symmetry condition that the quantities α_{2n+1} are real, while α_{2n} are purely imaginary.

Since $\Phi(t)$ and $\Psi(t)$ are regular in the entire lower half-plane, the equations obtained, if $F_1(t)$ and $F_2(t)$ are regarded as known functions, will represent the equations of the principal mixed problem of the theory of elasticity for a half-plane, and its solution can be written explicitly $(^1)$:

$$\Phi(z) = \frac{X(z)}{2\pi i} \int_{L_0} \frac{f(t)}{X^+(t)(t-z)}\, dt + C_0 X(z); \tag{7}$$

$$\Psi(z) = -\Phi(z) - \overline{\Phi(\bar{z})} - z\Phi'(z); \tag{8}$$

$$X(z) = (z+a)^{-1/2+i\beta}(z-a)^{-1/2-i\beta}, \quad \beta = (2\pi)^{-1}\ln\varkappa, \quad C_0 = (2\pi)^{-1}Pi;$$

$$f(t) = \begin{cases} F_1(t) & \text{on } L_0, \\ -F_2(t) & \text{on } L_0'. \end{cases}$$

Evaluating the integral entering into (7), we obtain, for points z lying in the lower half-plane:

$$\Phi(z) = X(z)\left\{C_0 \sum_{n=0}^{\infty} \alpha_n \left[\tilde{\gamma}_n(z) + (-1)^n G_n^*(z)\right] - \varepsilon_n \alpha_{-n} G_n(z)\right\}, \tag{9}$$

and, for points z lying in the upper half-plane:

$$\Phi(z) = X(z)\left\{C_0 + \sum_{n=0}^{\infty} \alpha_n \left[\tilde{G}_n(z) + (-1)^n \gamma_n^*(z)\right] - \varepsilon_n \alpha_{-n} \gamma_n(z)\right\}, \tag{10}$$

* The functions $F_1(t)$ and $F_2(t)$ are connected by the relation

$$F_1(t) + F_2(t) = -(\varkappa+1)\sum_{n=0}^{\infty} \alpha_n (t+hi)^{-n-1}.$$

where the notation has been introduced

$$\varepsilon_1 = 2, \quad \varepsilon_n = 1 \ (n \neq 1); \qquad G_n(z) = \sum_{r=0}^{n} \widetilde{T}_{0,r}^{(2)} (z-ih)^{r-n-1};$$

$$\widetilde{G}_n(z) = \sum_{r=0}^{n} T_{0,r}^{(2)} (z+ih)^{r-n-1}$$

(the definition of the numbers $T^{(2)}_{0,r}$ and $\widetilde{T}^{(2)}_{0,r}$ is given below);

$$v_n(z) = G_n(z) - \{X(z)(z-ih)^{n+1}\}^{-1}; \quad \tilde{v}_n(z) = \widetilde{G}_n(z) - \{X(z)(z+ih)^{n+1}\}^{-1}.$$

The sign $*$ here and below denotes a sum of the form

$$R^*_n(z) = (n+2)R_{n+2}(z) - 2ih(n+1)R_{n+1}(z) - nR_n(z).$$

3. To determine the quantities α, we shall use equation (3). Substituting into it the limiting values $\Phi_0(t)$ and $\Psi_0(t)$, taken from (4) and (5), and taking (8) into account, we obtain

$$\Phi(t) - (1+e^{-2i\varphi})\overline{\Phi(t)} + e^{-2i\varphi}\big[(t-\bar{t})\overline{\Phi'(t)} - \overline{\Phi(\bar{t})}\big] = \Omega(t) - \alpha_{-1}. \tag{11}$$

We now replace $\Phi(t)$ and $\overline{\Phi(t)}$ according to formulas (9) and (10). Multiplying both sides of the resulting equality by $\{2\pi i(t+hi)^k\}^{-1}$ ($k=-\infty,\dots,\infty$) and integrating over the contour L_1, we arrive at an infinite system of equations (in this connection it turns out that $\alpha_0 = 0$):

$$\alpha_k + \sum_{n=1}^{\infty}\alpha_n\big[\tilde{u}_{n,k} + k\{\tilde{u}^*_{n,k} + (-1)^n(\tilde{v}^*_{n,k} + k\{\overline{v}_{n,k}\})\}\big] -$$

$$-\sum_{n=2}^{\infty}\alpha_{-n}\big[\tilde{v}_{n,k} + (-1)^n k\{\overline{u}_{n,k}\}\big] = -C_0\big[k\{\overline{T}^{(1)}_{0,k-1}\} + \widetilde{T}^{(1)}_{0,k-1}\big] +$$

$$+2\alpha_{-1}\big[\tilde{v}_{n,k} - k\{\overline{u}_{n,k}\}\big], \tag{12}$$

$$\alpha_{-k} - \sum_{n=1}^{\infty}\alpha_n\big[v_{n,k} + (-1)^n u^*_{n,k}\big] + \sum_{n=2}^{\infty}\alpha_{-n}u_{n,k} = C_0 T^{(1)}_{0,k-1} - 2\alpha_{-1}u_{1,k},$$

where

$$u_{n,k} = \sum_{r=0}^{n}\widetilde{T}^{(2)}_{0,n-r}T^{(1)}_{r+1,k-1}; \qquad v_{n,k} = \sum_{r=0}^{n}T^{(2)}_{0,r}T^{(1)}_{0,k+n-r};$$

$$T^{(1)}_{n,k} = \frac{1}{k!}\left[\frac{X(t)}{(t-hi)^n}\right]^{(k)}_{t=-hi}; \qquad T^{(2)}_{n,k} = \frac{1}{k!}\left[\frac{1}{X(t)(t-hi)^n}\right]^{(k)}_{t=-hi}.$$

The quantities $\widetilde{T}^{(j)}$ are obtained from $T^{(j)}$ by replacing h by $-h$. In the definition of $\tilde{u}_{n,k}$ and $\tilde{v}_{n,k}$ the sign $\sim$ acts as a conjugation bar. The symbol $*$ has been explained above, and, finally, the braces $\{\ \}$ in all cases denote sums of the form

$$\{R_{n,k}\} = R_{n,k+2} + 2ihR_{n,k+1} - R_{n,k}.$$

The computation of the coefficients is considerably simplified if one uses the recurrence relations:

$$T^{(j)}_{r,k} = \frac{1}{2hi}(T^{(j)}_{r,k-1} - T^{(j)}_{r-1,k}), \qquad u_{n,k} = \frac{1}{2hi}(u_{n,k-1} - u_{n-1,k} - \widetilde{T}^{(2)}_{0,n} T^{(1)}_{0,k-1}),$$

as well as the easily proved equalities

$$\sum_{r=0}^{n+k} T^{(1)}_{0,r} T^{(2)}_{0,k+n-r} = \sum_{r=0}^{n+k} \widetilde{T}^{(1)}_{0,r} \widetilde{T}^{(2)}_{0,k+n-r} = 0.$$

The derivative functions $X(z)$ can be computed by the formula

$$X^{(n)}(z) = \frac{(-1)^n X(z) \sin \pi A}{\pi} \sum_{r=0}^{n} C^r_n \frac{\Gamma(\bar{A} + n - r)\Gamma(A + r)}{(z+a)^{n-r}(z-a)^r},$$

where $A = \dfrac{1}{2} + i\beta$. An analogous formula also holds for $\left[X^{-1}(z)\right]^{(n)}$.

It is easy to verify that all coefficients with indices n and k of the same parity are real, while those with indices of different parity are purely imaginary. Therefore the system of equations (12) is a system with real coefficients with respect to α_{2k+1} and $\operatorname{Im} \alpha_{2k}$.

In system (12), the terms containing α_{-1} have been moved to the right-hand side, because the equation of the second group for $k = 1$ becomes an identity. To determine α_{-1}, we use the second of equalities (4). If in it $\Omega(t)$ is replaced according to formula (11), then, after carrying out the necessary transformations, we arrive at the equation:

$$\alpha_{-1} = \Phi(-ih),$$

from which we determine α_{-1} after solving system (12).

System (12) will be quasiregular [4] for any values $h > 1$. To prove this it is sufficient to estimate the sum $\sum_{n=1}^{\infty} n|u_{n,k}|$, since the remaining sums are estimated analogously. Applying Cauchy' s inequalities to estimate the derivatives, we obtain

$$\sum_{n=1}^{\infty} n|u_{n,k}| < \frac{c(h)}{h^k}.$$

Estimates of the same kind are obtained also for the terms appearing on the right-hand sides of equations (12).

Using the properties of regular systems, one can show that $|\alpha_k|$ and $|\alpha_{-k}|$ decrease with increasing k no more slowly than kh^{-k}. Hence, incidentally, the uniform convergence of all the series introduced will follow.

To determine the pressure under the punch we use the formula

$$P(t) + iT(t) = -(Y_y - iX_y) = -(\kappa + 1)\left[\Phi(t) + \sum_{n=1}^{\infty} \frac{\alpha_n}{(t + hi)^{n+1}}\right],$$

where $\Phi(t)$ is defined by formula (9).

From the analysis of the solution obtained it is evident that the symmetry requirement was introduced only to simplify the algebraic transformations.

Moscow Institute of Railway Transport Engineers
named after I. V. Stalin

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Note: Figure translations are in progress. See original paper for figures.

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