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Abstract

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MATHEMATICS

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ON THE CONSTRUCTION OF FINITE-DIFFERENCE EQUATIONS IN THE APPROXIMATE SOLUTION OF LAPLACE' S EQUATION

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In a domain G, lying in a finite part of m-dimensional space and bounded by a surface S, one seeks a solution of Laplace' s equation

$$\Delta u = \sum_{\alpha=1}^{n} u_{x_\alpha x_\alpha} = 0 \tag{1}$$

with the Dirichlet boundary condition $u|_S = \varphi$.

Let $(S) \in \Pi_{k+1}(B, \lambda)$ $(k \geq 0, \lambda > 0)$ and $\varphi \in H(l, A, \lambda)$ $(0 \leq l \leq k+1)$ (^{1}).

Put $l+\lambda = \gamma$. To specify φ with accuracy ε it is necessary to know $H(\varepsilon) \preccurlyeq \varepsilon^{-\frac{m-1}{\gamma}}$ numbers (^{2}).

It is not difficult to verify that, under the usual (^{3}) method of solving Laplace' s equation by finite differences, in order to find a solution of equation (1) with accuracy ε one must solve a system of equations in not fewer than $\preccurlyeq H(\varepsilon)^{\frac{m}{m-1}}$ unknowns. In solving this system by the method of successive approximations, not fewer than $\preccurlyeq H(\varepsilon)^{\frac{m+2}{m-1}}$ arithmetic operations are performed, and by the method of successive over-relaxations not fewer than $\preccurlyeq H(\varepsilon)^{\frac{m+1}{m-1}}$ operations (^{4}).

Below, for $m = 2$ and $\gamma \leq 3$, a method of approximate solution will be considered in which the memory used does not exceed $\preccurlyeq H(\varepsilon)^{1+\beta}$ numbers, and the number of operations is not more than $\preccurlyeq H(\varepsilon)^{1+\frac{\gamma+\beta}{3}}$, where $\beta > 0$ is arbitrary.

On the basis of relations (8), (9), for any $\chi > 0$ one can indicate a method of solution with memory used less than $\preccurlyeq H(\varepsilon)^{1+\beta}$, where $\beta > 0$ is arbitrary, and with a number of operations less than $\preccurlyeq H(\varepsilon)^{1+\chi}$, under the assumption that there exist formulas analogous to formulas (3) and (4) (the sign of one of the coefficients is opposite to the sign of the others) of arbitrarily high order of accuracy.

It follows from the preceding arguments that the proposed method is, in a certain sense, unimprovable.

Under the assumptions made concerning (S) and φ, from (1) we have

$$u \in H(l, cA, \lambda') \quad \text{in } G, \tag{2}$$

where λ' is any number smaller than λ.

Consider the case $m = 2$ and $\gamma \leq 3$. Let $h > 0$ be some small number. Denote by $\overline{G}_s$ $(s \geq 2)$ the set of points of the squares

$$(q_\alpha - 1)2^{s-1}h \leq x_\alpha \leq (q_\alpha + 1)2^{s-1}h \quad (\alpha = 1, 2),$$

where q_α are arbitrary integers such that all interior points of the squares

$$(q_\alpha - 3)2^{s-1}h \leq x_\alpha \leq (q_\alpha + 3)2^{s-1}h \qquad (\alpha = 1, 2)$$

belong to G. Denote the set of interior points of $\overline{G}_s$ by G_s. Clearly, for $s \geq \log_2 \frac{D}{4h}$, where D is the diameter of the domain G, the sets G_s are empty.

Let σ be a number such that $0 < \sigma < 1$. Denote by P_s $(s = 2, 3, ...)$ the set of points $(\bar{x}_1, \bar{x}_2) \in \overline{G}_s - \overline{G}_{s+1}$ satisfying the condition $\{\bar{x}_1/2^{[\sigma s]}h\} = \{\bar{x}_2/2^{[\sigma s]}h\} = 0$; by P_1, the set of points $(\bar{x}_1, \bar{x}_2) \in G - \overline{G}_2$ such that $\{\bar{x}_1/h\} = \{\bar{x}_2/h\} = 0$, and all points of the squares $|x_\alpha - \bar{x}_\alpha| \leq h$ $(\alpha = 1, 2)$ belong to $\overline{G}$; by P_0, the set of all remaining points $(i_1 h, i_2 h) \in G - \overline{G}_2$ (i_α integers).

The points of the sets P_s $(s = 0, 1, 2, ...)$ will be called nodes and denoted by (i_1, i_2), where $i_\alpha = x_\alpha/h$ $(\alpha = 1, 2)$. In $(i_1, i_2) \in \overline{G}_s - \overline{G}_{s+1}$ $(s = 0, 1, 2, ...)$ set $\Delta_{i_1 i_2} = 2^{[\sigma s]}h$.

For solutions of equation (1) we have

$$l_\Delta(u(x_1, x_2)) = \frac{1}{20\Delta^2}\left[-20u(x_1, x_2) + 4\sum_{|j|+|k|=1} u(x_1 + j\Delta, x_2 + k\Delta) + \sum_{|j|,|k|=1} u(x_1 + j\Delta, x_2 + k\Delta)\right]$$

$$= O\left(\max_{|x_\alpha - \bar{x}_\alpha| < \Delta} \left|u^{\mu_1, \mu_2}_{x_1 x_2}\right|_{\bar{x}_1, \bar{x}_2} \Delta^6\right); \tag{3}$$

$$l^{x_1}_\Delta(u(x_1, x_2)) = \frac{1}{2048\Delta^2}\left[-2048u(x_1, x_2) + 784\sum_{|j|=1} u\left(x_1 + j\frac{\Delta}{2}, x_2\right)\right.$$

$$+109 \sum_{|j|,|k|=1} u\left(x_1 + j\frac{\Delta}{2}, x_2 + k\Delta\right) + 6 \sum_{|j|,|k|=1} u\left(x_1 + j\frac{\Delta}{2}, x_2 + k\,2\Delta\right)$$

$$+5 \sum_{|j|,|k|=1} u\left(x_1 + j\frac{3}{2}\Delta, x_2 + k\Delta\right)\Bigg] = O\left(\max_{\substack{|\bar{x}_1 - x_1| < (3/2)\Delta \\ |\bar{x}_2 - x_2| < 2\Delta}} \left|u^8_{x_1^{\mu_1} x_2^{\mu_2}}\right| \Delta^6\right). \quad (4)$$

For $i_1, i_2 \in P_0$ we replace $u_{x_\alpha x_\alpha}$ by divided differences in terms of the values $u_{i_1 i_2}$ at the nodes nearest to (i_1, i_2) and at boundary points lying on the straight lines $x_\alpha = i_\alpha h$ $(\alpha = 1, 2)$ $^{(3)}$. For $(i_1, i_2) \in P_s - \overline{G}_{s+1}$ $(s = 1, 2, \ldots)$ we set $l_{\Delta_{i_1 i_2}}(u_{i_1 i_2}) = 0$; for $(i_1, i_2) \in (P_s - P_{s+1}) \cap \overline{G}_{s+1}$ $(s = 1, 2, \ldots)$ we set $l^{x_1}_{\Delta_{i_1 i_2}}(u_{i_1 i_2}) = 0$ or $l^{x_2}_{\Delta_{i_1 i_2}}(u_{i_1 i_2}) = 0$, depending on which coordinate axis is parallel to the segment of the boundary of the domain G_{s+1} on which the given node lies.

Let

$$L_{i_1 i_2}(u_{i_1 i_2}) = 0 \quad (5)$$

be the system of the listed equations. By virtue of the maximum principle this system always has, and moreover has a unique, solution.

Define $W^r_{i_1 i_2}$ from the system of equations

$$W^r_{i_1 i_2}\big|_S = 0;$$

$$L_{i_1 i_2}(W^r_{i_1 i_2}) = \begin{cases} -1, & \text{if } \rho((i_1 h, i_2 h), S) \le rh, \\ 0, & \text{if } \rho((i_1 h, i_2 h), S) > rh. \end{cases}$$

As in (5), it is shown that

$$W^r_{i_1 i_2} \le C(B, h, \lambda) r^2 h^2. \quad (6)$$

We have (6)

$$\left|u_{x_1^{\mu_1} x_2^{\mu_2}}\right| \le \frac{\Phi}{\rho((x_1, x_2), S)^{k - \gamma'}} \quad \text{for } k > \gamma' = l + \lambda' \quad (7)$$

Using (6) and (7), we obtain

$$|u_{i_1 i_2} - u(i_1 h, i_2 h)| \le k h^{\gamma'} |\ln h| \quad \text{when } 6(1 - \sigma) = \gamma';$$

$$|u_{i_1 i_2} - u(i_1 h, i_2 h)| \leq K(6(\sigma - 1) + \gamma')h^{\min \gamma',\, 6(1-\sigma)} \quad \text{when } 6(1-\sigma) \neq \gamma'.$$

We solve system (5) with respect to the values $u_{i_1 i_2}$ corresponding to each equation, and solve it by the method of successive approximations. Using as a majorant the function $z_{i_1 i_2}$, determined from the system

$$z_{i_1 i_2}\big|_S = 0; \qquad L_{i_1 i_2}(z_{i_1 i_2}) = -\frac{1}{\Delta^2_{i_1 i_2}},$$

we obtain that, in order to determine the solution of system (5) with accuracy ε, it suffices to carry out

$$\asymp h^{2(\sigma-1)} \log \varepsilon$$

iterations.

The memory used in these computations is

$$\asymp \frac{1}{h}|\log h| \quad \text{when } \sigma = \frac{1}{2};$$

$$\asymp h^{\min -1,\, 2(\sigma-1)} \quad \text{when } \sigma \neq \frac{1}{2}.$$

Putting $\sigma = 1 - \gamma/6$, we obtain the result formulated above. Suppose now that we have at our disposal formulas analogous to formulas (3) and (4), of order of accuracy n, and, for $\gamma > 3$, also analogous formulas of accuracy $n - 2$ for approximating the solution of the equation near the boundary.

For $\gamma < n$, slightly modifying the construction of the domains G_s and carrying out similar arguments, we obtain, putting $\sigma = 1 - \gamma/n$, a method of solution with memory used less than

$$\asymp H(\varepsilon)^{r+\beta} \tag{8}$$

and with number of operations less than

$$\asymp H(\varepsilon)^{r + 2\frac{\gamma+\beta}{n(m-1)}},$$

where

$$r = \max\{1,\ \frac{m\gamma}{n(m-1)}\}; \qquad \beta > 0 \text{ is arbitrary.}$$

For $\lambda < 1,\ m = 2$ and $\lambda < 1,\ m = 3,\ k \geq l - 1$ in (2), one may put $\lambda' = \lambda$ (7). Assuming, when

$$1 - \frac{\gamma}{n} > \frac{1}{m}, \qquad \sigma = 1 - \frac{\gamma + \delta}{n},$$

where $\delta > 0$,

$$1 - \frac{\gamma + \delta}{n} > \frac{1}{m},$$

we obtain a method of solution with memory used

$$\asymp H(\varepsilon).$$

If system (5) is solved by the method of successive over-relaxations, then, apparently, the number of iterations will be

$$\asymp h^{\sigma - 1} \log \varepsilon,$$

and the number of operations no more than

$$\asymp H(\varepsilon)^{r + \frac{\gamma + \beta}{n(m-1)}},$$

where $\beta > 0$ is arbitrary.

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Note: Figure translations are in progress. See original paper for figures.