



---

Soviet-era science, translated into English

# Reports of the Academy of Sciences of the USSR

G. A. ZAITSEV

1957

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-195701.08108>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

## Abstract

## Full Text

Reports of the Academy of Sciences of the USSR  
1957. Volume 114, No. 1

## PHYSICS

G. A. ZAITSEV

# SHIFT OF THE ENERGY LEVELS OF A PARTICLE WITH SPIN 1/2 IN A COULOMB FIELD

(Presented by Academician N. N. Bogoliubov, 15 XII 1956)

Let a particle with spin 1/2 and negative charge (for example, a  $\mu^-$ -meson or an electron) be in a field with potential  $\varphi = Ze/r$ ,  $A_k = 0$ . We shall denote the charge of the particle by  $-e$  ( $e > 0$ ). Using the basic equation for a particle with spin 1/2 in an electromagnetic field, considered in <sup>(1)</sup>, we find the corresponding energy levels.

We shall assume that the particle is in a stationary state, so that

$$\hbar i \frac{\partial}{\partial t} \xi = E \xi, \quad \xi = [\exp(-iEt/\hbar)] \xi_0. \quad (1)$$

Equation (7) from <sup>(1)</sup> gives:

$$\hat{H} \xi_0 = 0,$$

$$\hat{H} = \frac{1}{r^2} \left( \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \right) + \frac{1}{\hbar^2 c^2} \left[ \left( E + \frac{Ze^2}{r} \right)^2 - m_0^2 c^4 \right] + \frac{\nabla_{\vartheta}^2 + iZ\alpha(1+\delta)\sigma_{(r)}}{r^2}, \quad (2)$$

where

$$\alpha = \frac{e^2}{\hbar c}, \quad \sigma_{(r)} = \frac{\sum_k x^k \sigma_k}{r}, \quad \nabla_{\vartheta}^2 = \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial}{\partial \vartheta} \right) + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2}.$$

We shall be able to achieve separation of variables if we put

$$\xi_0 = f(r) \begin{pmatrix} y_1(\vartheta, \varphi) \\ y_2(\vartheta, \varphi) \end{pmatrix},$$

where

$$- \left[ \nabla_{\vartheta\varphi}^2 + iZ\alpha(1 + \delta)\sigma_{(r)} \right] \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \hat{L}y = \lambda y. \quad (3)$$

Then the energy levels will be determined from the equation

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{\hbar^2 c^2} \left[ \left( E + \frac{Ze^2}{r} \right)^2 - m_0^2 c^4 \right] - \frac{\lambda}{r^2} \right\} f(r) = 0. \quad (4)$$

It is not difficult to verify that the operators

$$\hat{M}_k = \hbar \left( -i \sum_{j,s=1}^3 \varepsilon_{kjs} x^j \frac{\partial}{\partial x^s} + \frac{1}{2} \sigma_k \right) \quad (5)$$

commute with  $\hat{H}$  and  $\hat{L}$ .

Moreover, the operators  $\hat{L}$ ,  $\hat{M}_k$ , and  $\hat{M}^2 = \hat{M}_1^2 + \hat{M}_2^2 + \hat{M}_3^2$  commute with one another. Therefore we may choose  $y_1$  and  $y_2$  so that  $y$ , and consequently  $\xi_0$ , will simultaneously be an eigenfunction of the operators  $\hat{L}$ ,  $\hat{M}_3$ , and  $\hat{M}^2$ , i.e.

$$\hat{M}_3 y = \hbar m y, \quad (6)$$

$$\hat{M}^2 y = \hbar^2 j(j+1) y, \quad (7)$$

and the state of the particle will be characterized by the quantum numbers  $\lambda, m, j$  and by the radial quantum number, which we shall introduce below.

From the commutation relations between  $\hat{M}_k$ , as is known, it follows that for a given  $j$ ,  $m$  can take the values  $m = -j, -j+1, \dots, j-1, j$ . It is easy to verify that from conditions (6) and (7) it follows that  $y$  has the form

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \quad (8)$$

$$= \begin{pmatrix} \left( C_{j-\frac{1}{2}} P_{j-\frac{1}{2}, m-\frac{1}{2}} + C_{j+\frac{1}{2}} P_{j+\frac{1}{2}, m-\frac{1}{2}} \right) e^{i(m-\frac{1}{2})\varphi} \\ \left( -\sqrt{\frac{j-m}{j+m}} C_{j-\frac{1}{2}} P_{j-\frac{1}{2}, m+\frac{1}{2}} + \sqrt{\frac{j+m+1}{j-m+1}} C_{j+\frac{1}{2}} P_{j+\frac{1}{2}, m+\frac{1}{2}} \right) e^{i(m+\frac{1}{2})\varphi} \end{pmatrix},$$

where  $C_{j-\frac{1}{2}}, C_{j+\frac{1}{2}}$  are arbitrary constants;

$$P_{j-\frac{1}{2}, m-\frac{1}{2}} = \sqrt{j \frac{(j-m)!}{(j+m-1)!}} (1-x^2)^{\frac{m-\frac{1}{2}}{2}} \frac{1}{2^{j-\frac{1}{2}} (j-\frac{1}{2})!} \times \\ \times \frac{d^{j+m-1}}{dx^{j+m-1}} (x^2-1)^{j-\frac{1}{2}}, \quad x = \cos \vartheta. \quad (9)$$

Using the properties of the functions  $P$  (see, for example, (2), pp. 383-384), from (3) we find the ratio  $C_{j+\frac{1}{2}} : C_{j-\frac{1}{2}}$  and the third quantum number  $\lambda$ :

$$\lambda = (j + \frac{1}{2})^2 \mp \sqrt{(j + \frac{1}{2})^2 - \alpha^2(1 + \delta)^2} \quad (10)$$

(here and in the subsequent formulas, for simplicity, we put  $Z = 1$ ).

We see that the quantum number  $\lambda$ , for given  $j$  and  $m$ , can take only two values, corresponding to the two possible choices of sign in (10). If the quantity  $\alpha^2$  is neglected and in (10) one sets  $\alpha^2 = 0$ , then for this limiting case, instead of the quantum number  $\lambda$ , we may introduce another quantum number  $l$ , defined according to

$$\lambda = l(l+1), \quad l = j \mp \frac{1}{2}. \quad (11)$$

In order to preserve correspondence with the terminology adopted in spectroscopy, one may characterize the state of a particle with spin 1/2 by the number  $l$  (in addition to  $j$  and  $m$ ) also in the case when we do not neglect the quantity  $\alpha^2$ . To this end we put  $l = j - \frac{1}{2}$  or  $l = j + \frac{1}{2}$ , depending on the sign in (10). Then the quantum number  $\lambda$  may be expressed through  $j$  and  $l$  by the formula

$$\lambda = (j + \frac{1}{2})^2 + 2(l-j) \sqrt{(j + \frac{1}{2})^2 - \alpha^2(1 + \delta)^2}. \quad (12)$$

Now let us consider the equation for the radial function (4), writing it in the form:

$$\left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - A + \frac{2B}{r} - \frac{C}{r^2} \right) f(r) = 0, \quad (13)$$

where

$$A = \frac{m_0^2 c^2}{\hbar^2} \left[ 1 - \left( 1 + \frac{\mathcal{E}}{m_0 c^2} \right)^2 \right], \quad \mathcal{E} = E - m_0 c^2, \quad B = \frac{m_0 e^2}{\hbar^2} \left( 1 + \frac{\mathcal{E}}{m_0 c^2} \right),$$

$$C = \lambda - \alpha^2 = (j + \frac{1}{2})^2 \mp \sqrt{(j + \frac{1}{2})^2 - \alpha^2(1 + \delta)^2} - \alpha^2 = l'(l' + 1), \quad l' > 0. \quad (14)$$

The solution of this equation is well known (see, for example, (3), p. 438). In particular, in order to obtain discrete energy levels  $\mathcal{E}$ , it is necessary to introduce the radial quantum number  $k$  by the formula

$$B/\sqrt{A} = l' + k + 1. \quad (15)$$

$k$  can take the values  $k = 0, 1, 2, \dots$ . The energy levels are determined from the formula

$$1 + \frac{\mathcal{E}}{m_0 c^2} = \left[ 1 + \frac{\alpha^2}{(l' + k + 1)^2} \right]^{-1/2} = \left[ 1 + \frac{\alpha^2}{\left( k + \frac{1}{2} + \sqrt{\frac{1}{4} + \lambda - \alpha^2} \right)^2} \right]^{-1/2}. \quad (16)$$

Expanding the right-hand side of (16) in powers of  $\alpha^2$  and retaining terms of order  $\alpha^4$ , we shall have

$$\frac{\mathcal{E}}{h} = -\frac{R}{n^2} \left[ 1 + \frac{\alpha^2}{n^2} \left( \frac{n}{j + \frac{1}{2}} - \frac{3}{4} \right) + \frac{\alpha^2 (\delta + \frac{1}{2} \delta^2)}{n (j + \frac{1}{2}) (j + \frac{1}{2} \mp \frac{1}{2})} \right], \quad (17)$$

where

$$R = \frac{m_0 e^4}{4\pi \hbar^3} = \frac{\alpha^2 m_0 c^2}{2h}, \quad n = j + 1 \mp \frac{1}{2} + k = l + k + 1.$$

Thus, the deviation of the magnetic moment of the particle from the corresponding magneton, characterized by the quantity  $\delta$ , leads to a shift of the energy levels in comparison with the case when the Dirac equation is used without taking this deviation into account. In particular, the distance between the levels  $2^2 S_{1/2}$  and  $2^2 P_{1/2}$ , according to (17), is now given by the following formula:

$$h \left[ \nu(2^2 S_{1/2}) - \nu(2^2 P_{1/2}) \right] = -\frac{h}{3} R \alpha^2 \left( \delta + \frac{1}{2} \delta^2 \right). \quad (18)$$

Received  
28 X 1954

## REFERENCES

<sup>1</sup> G. A. Zaitsev, DAN, **113**, No. 6 (1957).

<sup>2</sup> H. Bethe, *Quantum Mechanics of the Simplest Systems*, 1935.

<sup>3</sup> A. A. Sokolov, D. D. Ivanenko, *Quantum Field Theory*, 1952.

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*