Quadrature-Free Nomography

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Abstract

Full Text

MATHEMATICS

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Let, for the given equation,

$$z = F(x, y), \qquad x_1 \leqslant x \leqslant x_2, \qquad y_1 \leqslant y \leqslant y_2, \tag{1}$$

which we shall sometimes write in the form $x = \psi(yz)$, it be possible to construct a working nomogram of aligned points, for which Massau' s equation has the form

$$\begin{vmatrix} f_1(x) & \varphi_1(x) \\ f_2(y) & \varphi_2(y) \\ f_3(z) & \varphi_3(z) \end{vmatrix} = 0. \tag{2}$$

It is required to find f_i and φ_i $(i = 1, 2, 3)$.

We regard x and z as independent variables, and y as their function. Equating the derivatives dy/dx, $\partial^2 y/\partial x^2$ obtained from equations (1) and (2), we eliminate $f_3(z)$ and $\varphi_3(z)$ from the resulting equalities and equation (2):

$$p(x, y) = -\frac{dy}{dx} = \frac{f_3 - f_2}{f_1 - f_3} \frac{f_1'(\varphi_2 - \varphi_1) - \varphi_1'(f_2 - f_1)}{f_2'(\varphi_2 - \varphi_1) - \varphi_2'(f_2 - f_1)}, \tag{3}$$

$$N(x, y) = -\frac{\partial^2 y}{\partial x^2} = p_x - p_y p = \left[\frac{\Delta_{1x}}{\Delta_1} - 2\frac{\Delta_{2x}}{\Delta_2} + p\left(\frac{\Delta_{2y}}{\Delta_2} - 2\frac{\Delta_{1y}}{\Delta_1}\right)\right] p, \tag{4}$$

where

$$\Delta_1 = f_1'(\varphi_2 - \varphi_1) - \varphi_1'(f_2 - f_1), \qquad \Delta_2 = f_2'(\varphi_2 - \varphi_1) - \varphi_2'(f_2 - f_1).$$

Equality (4) is an identity in x and y. To determine from it the functions f_1, φ_1 (or f_2, φ_2), it suffices to assign to the variable y (respectively x) 6 arbitrary values. This yields 6 algebraic equations for 6 unknown functions. In view of the assumption that equation (1) is nomographable, the system is consistent.

The functions found will depend on 36 constants (the values $f_2^{(i)}(y_k)$, $\varphi_2^{(i)}(y_k)$, $i = 0, 1, 2,\ k = 1, \dots, 6$). Eight constants may be chosen by making use of projective parameters.

When the functions f_1 and φ_1 are known, the determination of the remaining elements of the Massau determinant is carried out algebraically[(1)] (the solvability of equation (1) with respect to x is required). Finding the constants and checking the agreement of the constructed Massau equation with the original one decides the question of the nomographability of the latter.

From what has been set forth it is clear that, in order to clarify the question of the nomographability of a given equation, generally speaking, the existence of second derivatives of $F(x, y)$ is sufficient. Accordingly, to clarify the possibility of constructing a nomogram with one rectilinear scale, the existence of first derivatives is sufficient. Indeed, if equation (1) is reduced to the form

$$f_3 = \frac{f_1 - f_2}{\varphi_1 - \varphi_2},$$

then, equating the derivatives $\dfrac{\partial y}{\partial x}$, po

we obtain $\dfrac{\partial y}{\partial x} = -\dfrac{\Delta_1}{\Delta_2}$, or, in expanded form,

$$f_1'\varphi_2 - \varphi_1' f_2 + f_1 p\varphi_2' - \varphi_1 p f_2' + (\varphi_1' f_1 - \varphi_1 f_1') = p\left(\varphi_2' f_2 - \varphi_2 f_2'\right). \qquad (5)$$

Assigning to the variable y 5 different values, we obtain a system linear with respect to the unknowns $f_1', \varphi_1', f_1, \varphi_1, \varphi_1' f_1 - \varphi_1 f_1'$. By virtue of the assumption on nomographability, this system is consistent. Consequently,

$$f_1 = \frac{\begin{vmatrix} a_1 & b_1 & p(xy_1)(a_1d_1 - b_1c_1) & p(xy_1)d_1 & 1 \\ a_2 & b_2 & p(xy_2)(a_2d_2 - b_2c_2) & p(xy_2)d_2 & 1 \\ a_3 & b_3 & p(xy_3)(a_3d_3 - b_3c_3) & p(xy_3)d_3 & 1 \\ a_4 & b_4 & p(xy_4)(a_4d_4 - b_4c_4) & p(xy_4)d_4 & 1 \\ a_5 & b_5 & p(xy_5)(a_5d_5 - b_5c_5) & p(xy_5)d_5 & 1 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & p(xy_1)c_1 & p(xy_1)d_1 & 1 \\ a_2 & b_2 & p(xy_2)c_2 & p(xy_2)d_2 & 1 \\ a_3 & b_3 & p(xy_3)c_3 & p(xy_3)d_3 & 1 \\ a_4 & b_4 & p(xy_4)c_4 & p(xy_4)d_4 & 1 \\ a_5 & b_5 & p(xy_5)c_5 & p(xy_5)d_5 & 1 \end{vmatrix}}, \qquad (6)$$

$$\varphi_1 = \frac{\begin{vmatrix} a_1 & b_1 & p(xy_1)c_1 & p(xy_1)[a_1d_1 - b_1c_1] & 1 \\ a_2 & b_2 & p(xy_2)c_2 & p(xy_2)[a_2d_2 - b_2c_2] & 1 \\ a_3 & b_3 & p(xy_3)c_3 & p(xy_3)[a_3d_3 - b_3c_3] & 1 \\ a_4 & b_4 & p(xy_4)c_4 & p(xy_4)[a_4d_4 - b_4c_4] & 1 \\ a_5 & b_5 & p(xy_5)c_5 & p(xy_5)[a_5d_5 - b_5c_5] & 1 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & p(xy_1)c_1 & p(xy_1)d_1 & 1 \\ a_2 & b_2 & p(xy_2)c_2 & p(xy_2)d_2 & 1 \\ a_3 & b_3 & p(xy_3)c_3 & p(xy_3)d_3 & 1 \\ a_4 & b_4 & p(xy_4)c_4 & p(xy_4)d_4 & 1 \\ a_5 & b_5 & p(xy_5)c_5 & p(xy_5)d_5 & 1 \end{vmatrix}}. \tag{7}$$

If the substitution of particular values of y is replaced by differentiation with respect to y, followed by substitution of one and the same value $y = y_0$, then the number of constants entering the functions f_1, φ_1 will decrease (in the equation of the 6th order) to 16, and with the use of parameters of a projective transformation—to 8. These parameters can be found from the consistency conditions of the equations obtained with respect to the derivatives.

For equations of the 5th nomographic order with a rectilinear scale x, in the case of sufficient smoothness, we obtain

$$f_1(x) = \frac{\left(\frac{N}{p}\right)_{yy} - \left(\frac{N}{p}\right)_y a - 2bp_y + pab - pd}{2\left(\frac{N}{p}\right)_y - p_{yy} - p_{ya} - pc - pb + pa^2} \quad \text{for } y = y_0. \tag{8}$$

The parameters a, b, c, d can be determined from the consistency condition, with respect to the derivatives f_1', f_1'', of the equations

$$\frac{N}{p} = \frac{f_1''}{f_1'} - pf_1, \tag{9}$$

$$\left(\frac{N}{p}\right)_y = 2f_1' - p_y f_1 - p\left(f_1^2 + af_1 - b\right) \tag{10}$$

and equation (8).

The scale z is determined from the equation

$$\varphi_3(z) = \frac{-p[\psi(y_0z), y_0]}{f_1'[\psi(y_0z)]}, \qquad f_3(z) = \varphi_3(z) f_1[\psi(y_0z)].$$

For equations of the 4th nomographic order, the method set forth is very effective.

Theorem. If, for equation (1), a nomogram of the 4th order (Cauchy or Clark— indifferently) can be constructed, then the corresponding anamorphosis is written in the form

$$u = \frac{\begin{vmatrix} \dfrac{F_y(xy_3)}{F_x(xy_3)} & \dfrac{F_y(xy_2)}{F_x(xy_2)} \\ \dfrac{\psi_y(y_3z)}{\psi_z(y_3z)} & \dfrac{\psi_y(y_2z)}{\psi_z(y_2z)} \end{vmatrix}}{\begin{vmatrix} \dfrac{F_y(xy_1)}{F_x(xy_1)} & \dfrac{F_y(xy_2)}{F_x(xy_2)} \\ \dfrac{\psi_y(y_1z)}{\psi_z(y_1z)} & \dfrac{\psi_y(y_2z)}{\psi_z(y_2z)} \end{vmatrix}}, \qquad v = \frac{\begin{vmatrix} \dfrac{F_y(xy_1)}{F_x(xy_1)} & \dfrac{F_y(xy_3)}{F_x(xy_3)} \\ \dfrac{\psi_y(y_1z)}{\psi_z(y_1z)} & \dfrac{\psi_y(y_3z)}{\psi_z(y_3z)} \end{vmatrix}}{\begin{vmatrix} \dfrac{F_y(xy_1)}{F_x(xy_1)} & \dfrac{F_y(xy_2)}{F_x(xy_2)} \\ \dfrac{\psi_y(y_1z)}{\psi_x(y_1z)} & \dfrac{\psi_y(y_2z)}{\psi_z(y_2z)} \end{vmatrix}}.$$

If equation (1) is of the 3rd nomographic order, then

$$\frac{F_y(xy)}{F_x(xy)} = \alpha(x)\,\beta(y), \qquad \frac{\psi_y(yz)}{\psi_z(yz)} = \gamma(z)\,\beta(y),$$

and the anamorphosis becomes indeterminate.

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CITED LITERATURE

1. G. E. James-Levy, *Uch. zap. MGU*, No. 163 (1953).

Note: Figure translations are in progress. See original paper for figures.