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**Abstract**

**Full Text**

**MATHEMATICS**

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## **ON THE DIFFERENTIABILITY OF SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS**

*(Presented by Academician S. L. Sobolev on 22 IX 1956)*

Let  $\Omega$  be an open bounded domain of variation of the variable  $x(x_1, x_2, \dots, x_n)$ , and let  $\Gamma$  be the boundary of this domain. We shall require that the function defining  $\Gamma$  in local coordinates be twice continuously differentiable.

Consider the elliptic differential equation

$$L(u) = \sum_{i,k=1}^n \frac{\partial}{\partial x_i} \left( a_{ik}(x) \frac{\partial u}{\partial x_k} \right) = f(x), \quad (1)$$

which is to be integrated inside  $\Omega$  under the homogeneous boundary condition

$$u|_{\Gamma} = 0. \quad (2)$$

We assume that the coefficients  $a_{ik}(x)$  are twice continuously differentiable in the closed domain  $\Omega + \Gamma$ , and that the quadratic form composed of the coefficients of equation (1) satisfies the inequality

$$\sum_{i,k=1}^n a_{ik} \xi_i \xi_k \geq \mu \sum_{i=1}^n \xi_i^2,$$

where  $\mu$  is a positive constant.

It is known <sup>(1)</sup> that if  $f$  satisfies a Lipschitz condition with exponent  $\alpha$  ( $0 < \alpha < 1$ ) in  $\Omega + \Gamma$ , then the second derivatives of the solution of problem (1)–(2) also possess this property. In the work <sup>(2)</sup> of O. A. Ladyzhenskaya it was proved that if  $f \in L_2(\Omega)$ , then the second generalized (in the sense of S. L. Sobolev) derivatives of the solution belong to  $L_2(\Omega)$ .\* We are interested in the analogous question for the case  $f \in L_p(\Omega)$  ( $p > n/2$ ).

By a generalized solution of problem (1)–(2) we shall mean a function  $u \in W_p^{(2)}(\Omega)$  which: 1) almost everywhere in  $\Omega$  satisfies equation (1), and 2) almost everywhere on the boundary  $\Gamma$  vanishes.

**Lemma 1.** *The linear elliptic equation with constant coefficients*

$$\sum_{i,k=1}^n \bar{a}_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{j=1}^n \bar{b}_j \frac{\partial u}{\partial x_j} = f, \quad f \in L_p(\Omega) \quad (p > 1) \quad (3)$$

has a generalized solution which vanishes on the ellipsoid

$$\sum_{i,k=1}^n \bar{a}_{ik} x_i x_k = r^2 \quad (r \text{ constant}).$$

\* Following S. L. Sobolev <sup>(3)</sup>, spaces of functions having generalized derivatives of order  $l$  in  $\Omega$ , summable to the power  $p$ , will be denoted below by  $W_p^{(l)}$ . The notation of the norms corresponds to the notation in <sup>(3)</sup>.

**Proof.** The equation under consideration, by means of a suitable change of variables and of the function, can be reduced to the canonical form

$$\Delta u - a^2 u = f.$$

The existence of a solution of the latter equation which vanishes on the sphere of radius  $r$  can be proved in exactly the same way as in <sup>(4)</sup>, by means of the transformation indicated by S. G. Mikhlin <sup>(5)</sup>. Moreover, for a solution of equation (3) satisfying the indicated boundary condition, the estimate

$$\|u\|_{W_p^{(2)}(\Omega)} \leq C \|f\|_{L_p(\Omega)}$$

will hold, where  $C$  is a constant independent of  $u$  and  $f$ .

**Lemma 2.** If there exists a continuous solution  $u(x)$  of problem (1)–(2) and  $f \in L_p(\Omega)$  for  $p > n/2$ , then the estimate

$$\|u\|_C \leq A \|f\|_{L_p},$$

holds, where  $C$  is a constant independent of  $u$ ,  $f$ .

**Proof.** As is known <sup>(6)</sup>, the solution  $u(x)$  can be written in the form

$$u(x) = \int_{\Omega} \Gamma(x, \xi) f(\xi) d\Omega,$$

where the integration is performed with respect to the variable point  $\xi$ . The function  $\Gamma(x, \xi)$  has a singularity at  $x = \xi$  of order  $\rho^{2-n}$ , where  $\rho$  is the distance between  $x$  and  $\xi$ . By S. L. Sobolev's theorem on potentials <sup>(3)</sup>, the function

$$\varphi(x) = \int_{\Omega} f(\xi) \frac{d\Omega}{\rho^\lambda}$$

will be continuous for  $\lambda < n/p'$  ( $1/p + 1/p' = 1$ ). Substituting  $\lambda = n - 2$ , we obtain  $p > n/2$ .

**Lemma 3.** If  $u(x)$  is a solution of problem (1)–(2) twice continuously differentiable inside  $\Omega$ , and  $f$  is a continuous function, then for  $p > n/2$  the inequality

$$\int_{\Omega'} \left| \frac{\partial^2 u}{\partial x_i \partial x_k} \right|^p d\Omega \leq B \int_{\Omega} |f|^p d\Omega \quad (i, k = 1, 2, \dots, n),$$

holds, where  $B$  is a constant independent of  $u, f$ , and  $\Omega'$  is any interior subdomain of the domain  $\Omega$ .

**Proof.** It suffices to prove the lemma for a rectangular parallelepiped entirely contained inside  $\Omega'$ . Take a point  $P_0$  lying inside  $\Omega'$ , and construct a cube with edge  $\delta$  and center  $P_0$ . Choose  $\delta$  so that the lateral surface of the cube is at a distance greater than  $d/2$  from  $\Gamma$ . Consider the ellipsoid

$$\sum_{i,k=1}^n a_{ik}(P_0)(x_i - x_i^0)(x_k - x_k^0) = r^2 \quad (P_0 = P_0(x_1^0, \dots, x_n^0)).$$

Since the eigenvalues of the matrix  $\|a_{ik}\|$  are strictly positive, the number  $r$  can be chosen so that the ellipsoid  $T_0$ , lying outside  $Q_0$  and inside  $\Omega'$ , has small diameter when  $\delta$  is small.

By the uniqueness of the solution of problem (1)–(2), the function  $u(x)$  will satisfy equation (1) inside the ellipsoid, and on the boundary  $T_0$  the condition  $u|_{T_0} = u$ . The function  $u$  can be decomposed into two summands

$$u = v + w, \tag{4}$$

where

$$L(v) = 0, \quad v|_{T_0} = u; \quad L(w) = f, \quad w|_{T_0} = 0. \tag{5}$$

The function  $v(x)$ , as a solution of the homogeneous equation (1), can be represented in the form of the surface integral (6)

$$v(x) = \int_{T_0} R(x, \xi) u(\xi) d\xi,$$

where  $\xi$  lies on  $T_0$ , and  $R$  is continuous for  $x \neq \xi$ . Therefore

$$\int_{Q_0} \left| \frac{\partial^2 v}{\partial x_i \partial x_k} \right|^p d\Omega \leq C \max_{x \in \Omega} |u(x)|. \quad (6)$$

Equation (1) for the function  $w(x)$  can be rewritten as follows:

$$\sum_{i,k=1}^n a_{ik}(x) \frac{\partial^2 w}{\partial x_i \partial x_k} + \sum_{i,k=1}^n \frac{\partial a_{ik}}{\partial x_i} \frac{\partial w}{\partial x_k} = f(x).$$

Consider inside  $T_0$  the approximate equation with constant coefficients

$$\sum_{i,k=1}^n a_{ik}(P_0) \frac{\partial^2 \tilde{w}}{\partial x_i \partial x_k} + \sum_{i,k=1}^n \frac{\partial a_{ik}(P_0)}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_k} = f(x)$$

and the same boundary condition for  $\tilde{w}$ . The function  $\tilde{w}(x)$ , which vanishes on  $T_0$ , satisfies the conditions of Lemma 1. Therefore

$$\int_{\tilde{K}_0} \left| \frac{\partial^2 \tilde{w}}{\partial x_i \partial x_k} \right|^p d\Omega \leq C \int_{\tilde{K}_0} |f|^p d\Omega,$$

where  $K_0$  is the interior of the ellipsoid  $T_0$ . Since the diameter of  $K_0$  is small together with  $\delta$ , and the coefficients are continuously differentiable, it follows that

$$\int_{K_0} \left| \frac{\partial^2 w}{\partial x_i \partial x_k} \right|^p d\Omega \leq C \int_{K_0} |f|^p d\Omega.$$

Since  $Q_0 \subset K_0$ , strengthening the inequality, we obtain

$$\int_{Q_0} \left| \frac{\partial^2 w}{\partial x_i \partial x_k} \right|^p d\Omega \leq C \int_{\Omega} |f|^p d\Omega. \quad (7)$$

Inequalities (6) and (7) give

$$\int_{Q_0} \left| \frac{\partial^2 u}{\partial x_i \partial x_k} \right|^p d\Omega \leq C \left( \int_{\Omega} |f|^p d\Omega + \max_{x \in \Omega} |u(x)| \right).$$

Hence, and from Lemma 2, for  $p > n/2$  we obtain the inequality

$$\int_{Q_0} \left| \frac{\partial^2 u}{\partial x_i \partial x_k} \right|^p d\Omega \leq C \int_{\Omega} |f|^p d\Omega,$$

which was to be proved.

We shall say that a domain  $G' \subset G$  rests on  $\Gamma$  if the intersection of the boundary of  $G'$  with  $\Gamma$  is an  $(n-1)$ -dimensional manifold of nonzero measure.

**Lemma 4.** *Suppose that the conditions of Lemma 3 are satisfied. Then*

$$\int_{G'} \left| \frac{\partial^2 u}{\partial x_i \partial x_k} \right|^p d\Omega \leq B \int_{\Omega} |f|^p d\Omega,$$

where  $G'$  rests on  $\Gamma$ .

**Proof.** Without loss of generality, one may assume that the common part of the boundary  $G'$  and  $\Gamma$  is a piece of the plane  $x_n = 0$ . Take inside  $\overline{G'} \cap \Gamma$  a square of sufficiently small diameter  $\delta$ , and construct the cube  $Q_\delta$  resting on the indicated square. Denote all faces of this cube by  $Q'_\delta$ . Inside  $Q_\delta$ , take a domain  $G_1$  resting on  $x_n = 0$ . Again represent the function  $u$  in the form of a sum of functions  $u = v + w$  satisfying equations (4) and (5) and the boundary conditions

$$v|_{Q'_\delta} = u, \quad w|_{Q'_\delta} = 0.$$

Using the smallness of  $\delta$ , it is easy to show\* that

$$\int_{Q_\delta} \left| \frac{\partial^2 w}{\partial x_i \partial x_k} \right|^p d\Omega \leq C \int_{Q_\delta} |f|^p d\Omega$$

and, a fortiori,

$$\int_{G_1} \left| \frac{\partial^2 w}{\partial x_i \partial x_k} \right|^p d\Omega \leq C \int_{\Omega} |f|^p d\Omega.$$

Schauder <sup>(1)</sup> showed that

$$\max_{x \in G_1} \left| \frac{\partial^2 v}{\partial x_i \partial x_k} \right| \leq C \max_{x \in \Omega} |u|.$$

From this it is easy to obtain that

$$\int_{G_1} \left| \frac{\partial^2 v}{\partial x_i \partial x_k} \right|^p d\Omega \leq C \int_{\Omega} |f|^p d\Omega.$$

Then

$$\int_{G_1} \left| \frac{\partial^2 u}{\partial x_i \partial x_k} \right|^p d\Omega \leq C \int_{\Omega} |f|^p d\Omega,$$

as was required to be proved.

From Lemmas 3 and 4 it follows that, for a twice differentiable solution of problem (1)–(2), the estimate

$$\int_{\Omega} \left| \frac{\partial^2 u}{\partial x_i \partial x_k} \right|^p d\Omega \leq C \int_{\Omega} |f|^p d\Omega.$$

is valid.

From this inequality, by means of the usual arguments, one obtains the theorem:

**Theorem.** If  $a_{ik}(x)$  are continuously differentiable in  $\Omega + \Gamma$  and  $f \in L_p(\Omega)$  ( $p > n/2$ ), then there exists a generalized solution of problem (1)–(2) satisfying the inequality

$$\|u\|_{W_p^{(2)}(\Omega)} \leq C \|f\|_{L_p(\Omega)},$$

where  $C$  is a constant independent of  $u$ ,  $f$ .

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- <sup>3</sup> S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, 1950.
- <sup>4</sup> A. I. Koshelev, *Matem. sborn.*, **32** (74), 3 (1953).
- <sup>5</sup> S. G. Mikhlin, *DAN*, **78**, No. 3 (1951).
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\* In work <sup>(4)</sup> we showed that if  $L(u) = \Delta u$ , then Lemma 1 will also be valid when  $\Omega$  is a square. It is not difficult to extend this result to the case of an arbitrary number of variables and then to argue in the same way as in the proof of Lemmas 1 and 3.

*Note: Figure translations are in progress. See original paper for figures.*

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