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N. N. VAKHANIYA

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Abstract

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N. N. VAKHANIYA

ON A BOUNDARY-VALUE PROBLEM WITH DATA ON THE ENTIRE BOUNDARY FOR A HYPERBOLIC SYSTEM EQUIVALENT TO THE EQUATION OF VIBRATION OF A STRING

(Presented by Academician S. L. Sobolev on 7 V 1957)

We consider the following problem, proposed by S. L. Sobolev: in the rectangle $R: \ 0 \leq x \leq X, \ 0 \leq t \leq T$, with boundary Γ and ratio of side lengths $T/X = \rho$, find a solution of the system

$$\begin{aligned} \frac{\partial u_1}{\partial x} &= \frac{\partial u_2}{\partial t}, \\ \frac{\partial u_1}{\partial t} &= \frac{\partial u_2}{\partial x} \end{aligned} \qquad (1)$$

with the boundary condition

$$a u_1\big|_\Gamma + b u_2\big|_\Gamma = f; \qquad (2)$$

a, b, and f are prescribed on Γ.

S. L. Sobolev, in the note ([1]), investigated this problem for a square ($\rho = 1$). S. L. Sobolev' s method can be extended to a rectangle with arbitrary rational ρ. This is done in the first part of the present note.* In the second part the case of irrational ρ is considered.

1. $\rho = m/n$, where m and n are relatively prime positive integers. Without loss of generality one may assume that $T = m$, $X = n$; a, b, and f are assumed to be continuous functions of the arc length s, measured from the origin of the coordinates. Divide Γ into unit intervals $I_1, I_2, \dots, I_{2(m+n)}$, where I_k ($k = 1, 2, \dots, 2(m+n)$) is the interval $k-1 < s < k$. It is known ([2]) that the path of a light ray issuing from any point of any of the intervals I_k along one of the characteristic directions of system (1) (the lines $x \pm t = \text{const}$), with elastic reflection at the boundary, is closed in the case of any rational ρ. Further, one can prove that: a) into each interval

I_k there falls one node (point of reflection) of the closed light path; thus the number of nodes is $2(m+n)$; b) if the first node (the node in I_1) has the value of the parameter s equal to ξ, then the k-th node has the value of s equal to $k-1+\xi$ if k is odd, and $k-\xi$ if k is even.

Taking these properties into account and repeating the arguments from ([1]), it is not difficult to show that the question of solvability of the problem posed reduces to the question of the existence of zeros in the interval $0<\xi<1$ of the following determinant of order $2(m+n)$:

$$\Delta_{mn}(\xi)=(-1)^l\{\alpha_1\alpha_3\cdots\alpha_{2(m+n)-1}\beta_2\beta_4\cdots\beta_{2(m+n)}-\alpha_2\alpha_4\cdots\alpha_{2(m+n)}\beta_1\beta_3\cdots\beta_{2(m+n)-1}\},$$

where

$$\alpha_k=a(k-1+\xi)+b(k-1+\xi),\quad \beta_k=a(k-1+\xi)-b(k-1+\xi),$$

when k is odd $(k=1,3,\dots,2(m+n)-1)$, and

$$\alpha_k=a(k-\xi)+b(k-\xi),$$

* This case was also considered by Yu. Radvigin.

$\beta_k=a(k-\xi)-b(k-\xi)$, when k is even $(k=2,4,\dots,2(m+n))$; the parity of l depends on the parities of m and n. $\Delta_{11}(\xi)$ coincides with the determinant in (1). If, instead of $\Delta_{11}(\xi)$, one everywhere takes $\Delta_{mn}(\xi)$, then all the conclusions from (1) remain valid. In particular, the following theorem holds.

Theorem 1. *If $\Delta_{mn}(\xi)\neq 0$ for all ξ in the interval $0<\xi<1$, then there exists a unique solution of the posed problem.*

2. Let ρ be irrational. We shall assume that $a^2-b^2\neq 0$ everywhere on Γ. Then, obviously, we may suppose that everywhere on Γ

$$a^2-b^2=1. \tag{3}$$

We first prove the uniqueness theorem.

Theorem 2. *Let ρ be any irrational number; let $u_1(x,t)$, $u_2(x,t)$ be any (continuous) solution of system (1). If $f\equiv 0$ on Γ and $u_2(x,t)$ vanishes at some point M of the boundary Γ, then $u_1(x,t)\equiv 0$ and $u_2(x,t)\equiv 0$ in R.*

Proof. The general solution of system (1) has the form

$$u_1(x,t)=\varphi(x+t)+\psi(x-t),\qquad u_2(x,t)=\varphi(x+t)-\psi(x-t),$$

where the function $\varphi(\xi)$ is defined on the interval $0 \leq \xi \leq X+T$, and $\psi(\xi)$ on the interval $-T \leq \xi \leq X$. In terms of φ and ψ, the boundary condition (2) takes the form ($f \equiv 0$)

$$(a+b)\varphi(x+t)\big|_\Gamma + (a-b)\psi(x-t)\big|_\Gamma = 0. \tag{2'}$$

Draw through the point M one of the characteristics of system (1), for example the line $x+t = \text{const}$. Denote by M_1 the point of intersection of this line with Γ. From M_1 draw the characteristic $x-t = \text{const}$, and denote the point of intersection by M_2. From M_2 draw the characteristic $x+t = \text{const}$, denoting the point of intersection by M_3. Drawing alternately the characteristics $x-t = \text{const}$ and $x+t = \text{const}$, and continuing this process indefinitely, we obtain a set of points $M, M_1, M_2, \ldots$ everywhere dense(2) on Γ. It is not difficult to see that, by virtue of (2), (2′), and (3), φ and ψ are zero on this set and, in view of continuity, are identically zero on Γ. But this means that $\varphi(\xi) \equiv 0$ for all $\xi \in [0, X+T]$ and $\psi(\xi) \equiv 0$ for all $\xi \in [-T, X]$. Thus $u_1(x,t) \equiv 0$ and $u_2(x,t) \equiv 0$.

Remark. Taking $a = 1$, $b = 0$, we obtain the uniqueness theorem for the Dirichlet problem for the equation of vibrations of a string. Obviously, in this case the requirement that the function u_2 vanish at a boundary point becomes superfluous. This requirement also becomes superfluous in some other cases, for example, in the case when $a^2 - b^2 = 0$ on any finite nonempty set of points on Γ, but $a^2 + b^2 \neq 0$ on Γ.

Passing to the question of existence of a solution, suppose* that the irrational number ρ is not too rapidly approximated by rational numbers, more precisely

$$\left|\rho - \frac{m}{n}\right| > \frac{A}{n^{K+1}}, \tag{4}$$

where A is a fixed positive constant; K is a fixed positive integer; m and n are arbitrary integers. We prove an existence theorem.

Theorem 3. *Suppose: a) ρ satisfies inequality (4) for some K; a, b satisfy condition (3); b) on each side of the rectangle R, $f \in C^{K+4}$, $\ln|a+b| \in C^{2K+6}$, and these functions and their even derivatives up to orders respectively $2\left[\frac{K+4}{2}\right]$ and $2K+6$ are continuous*

* The set of numbers ρ which, for an arbitrarily fixed K, do not satisfy this requirement has Lebesgue measure zero. In particular, algebraic numbers of degree $K+1$ satisfy this assumption.

at the four corner points of the rectangle. Under these assumptions there exists a twice continuously differentiable solution of the system (1), satisfying the boundary condition (2).

Proof. For brevity, let us call a pair of functions satisfying the system (1) **hyperbolically conjugate functions**. It is easy to verify the validity of the following elementary properties of hyperbolically conjugate functions: a) each member of a pair of hyperbolically conjugate functions is determined by the other up to an arbitrary constant summand; b) inside the rectangle R both hyperbolically conjugate functions have the same degree of smoothness, and if one of them is twice differentiable, then each of them satisfies the equation of vibration of a string; c) if v_1, v_2 and w_1, w_2 are two pairs of hyperbolically conjugate functions, then $v_1 w_1 + v_2 w_2$ and $v_1 w_2 + v_2 w_1$ are also hyperbolically conjugate functions. It is not difficult to see that these properties of hyperbolically conjugate functions make it possible to reduce the boundary-value problem (2) for the system (1) to the Dirichlet problem* for the equation of vibration of a string in the special particular case when the functions a and b are boundary values on Γ of functions hyperbolically conjugate in the domain R. But it is clear that the problem of such a continuous continuation of the functions a and b into the interior of the rectangle is overdetermined and, generally speaking, has no solution. However, by multiplying the boundary condition (2) by a suitably chosen function, one can ensure that the new coefficients at $u_1|_\Gamma$ and $u_2|_\Gamma$ in relation (2) already have the required special property. To find such a "linking" function, the solution of the Dirichlet problem for the equation of vibration of a string will be needed. We give the exact proof.

Let $\Theta(x,t)$ be the solution of the Dirichlet problem for the equation of vibration of a string with boundary condition

$$\Theta|_\Gamma = \operatorname{ar\,th} \frac{b}{a} = \ln|a+b|.$$

Taking into account the result concerning the conditions for existence of a solution of the Dirichlet problem for the equation of vibration of a string ([2]), we may assert that, under our assumptions,** the function $\Theta(x,t)$ does indeed exist and belongs to C^{K+4}. Denote by $\Lambda(x,t)$ the function hyperbolically conjugate to $\Theta(x,t)$, and by λ the boundary value on Γ of the function $\Lambda(x,t)$. It can be shown that on each side of the rectangle R, $\lambda \in C^{K+4}$ and, moreover, λ itself and all its even derivatives up to order

$$2\left[\frac{K+4}{2}\right]$$

are continuous at the four corner points of the rectangle R. By assumption, the function f also has this property, so that the function $e^\lambda f$ will likewise have this property. Therefore there exists a twice differentiable function $F_1(x,t)$ which is a solution of the Dirichlet problem for the equation of vibration of a string with boundary function

$$F_1|_\Gamma = e^\lambda f.$$

It is easy to see that, in view of condition (3), $\operatorname{ch}\Theta|_\Gamma = a$ and $\operatorname{sh}\Theta|_\Gamma = b$. Therefore, if u_1 and u_2 satisfy the system (1) and the boundary condition (2), then the function

$$e^{\Lambda}\operatorname{ch}\Theta\, u_1 + e^{\Lambda}\operatorname{sh}\Theta\, u_2,$$

which, as can be verified directly, satisfies the equation of vibration of a string, also assumes on Γ the boundary function $e^{\lambda} f$. Hence, by the uniqueness of the solution of the Dirichlet problem for the equation of vibration of a string, we have

$$e^{\Lambda}\operatorname{ch}\Theta\, u_1 + e^{\Lambda}\operatorname{sh}\Theta\, u_2 = F_1(x,t).$$

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* The idea of reducing to the Dirichlet problem is borrowed from the theory of boundary-value problems for analytic functions ([3]).

** In the work ([4]) it is additionally required that at the four corner points of R the boundary function and all its even derivatives up to the corresponding order vanish. However, one can dispense with this restriction by adding to the sought solution a completely definite particular solution which, as is easily shown, can always be constructed.

Further, one can verify directly that the function $e^{\Lambda}\operatorname{sh}\Theta\, u_1 + e^{\Lambda}\operatorname{ch}\Theta\, u_2$ is the hyperbolically conjugate function to $F_1(x,t)$ (thus, the role of the "linking" function is played by the function e^{λ}). Thus, denoting by $F_2(x,t)$ the function conjugate to $F_1(x,t)$, we have

$$e^{\Lambda}\operatorname{sh}\Theta\, u_1 + e^{\Lambda}\operatorname{ch}\Theta\, u_2 = F_2(x,t) + B,$$

where B is an arbitrary constant. From the last two equalities we find the solution $u_1(x,t), u_2(x,t)$

$$u_1(x,t) = e^{-\Lambda}\{F_1\operatorname{ch}\Theta - F_2\operatorname{sh}\Theta - B\operatorname{sh}\Theta\},$$

$$u_2(x,t) = e^{-\Lambda}\{F_2\operatorname{ch}\Theta - F_1\operatorname{sh}\Theta + B\operatorname{ch}\Theta\}.$$

In accordance with the uniqueness theorem, the solution is determined up to an arbitrary constant B, which can be found by prescribing u_2 at some point of the boundary $(\operatorname{ch}\Theta)|_{\Gamma} = a \neq 0$ everywhere by virtue of (3)).

Remark. If on each side of the rectangle the functions f and $\ln|a+b|$ are trigonometric polynomials, then the problem is solvable for any irrational ρ.

Remark. It is not difficult to see that the solution u_1, u_2 depends continuously on the functions f and $\ln|a+b|$ in the sense of the uniform metric, respectively of type $(K+4,2)$ and $(2K+6,2)$.

Tbilisi Mathematical Institute named after A. M. Razmadze
Academy of Sciences of the Georgian SSR

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Note: Figure translations are in progress. See original paper for figures.