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Abstract

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ON STRONGLY MINIMAL SURFACES OF A RIEMANNIAN SPACE

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1°. Various connections of two-dimensional minimal surfaces with complex-analytic surfaces were noted by Schwarz ([1]), Eisenhart ([2]), Borůvka ([3]), and others. However, their results cannot be transferred to arbitrary minimal surfaces with a number of dimensions greater than two. In the present note a class of minimal surfaces is introduced which are as closely connected with complex-analytic surfaces of the corresponding number of dimensions as two-dimensional minimal surfaces are with complex-analytic surfaces of two real dimensions.

We shall call a surface of $2n$ dimensions in an N-dimensional Riemannian space a **strongly minimal surface** if there exists for it a frame of the first order in which its second fundamental object Λ^{ξ}_{pq} ([4]) and metric tensor g_{pq} satisfy the equalities

$$\begin{aligned} \Lambda^{\xi}_{ij} &= -\Lambda^{\xi}_{n+i\,n+j}, \qquad \Lambda^{\xi}_{n+i\,j} = \Lambda^{\xi}_{i\,n+j}, \\ g_{ij} &= g_{n+i\,n+j}, \qquad g_{i\,n+j} = -g_{n+i\,j}, \end{aligned} \tag{1}$$

$$i = 1, \dots, n, \qquad \xi = 2n+1, \dots, N, \qquad p = 1, \dots, 2n.$$

2°. Theorem 1. *In order that a $2n$-dimensional surface of an N-dimensional Riemannian space be strongly minimal, it is necessary that the equalities*

$$A^{\xi_1 \dots \xi_{2k-1}} \equiv \Lambda^{p_1 \xi_1}_{[p_1} \Lambda^{p_2 \xi_2}_{p_2} \cdots \Lambda^{p_{2k-1} \xi_{2k-1}}_{p_{2k-1}]} = 0 \qquad (k = 1, \dots, n). \tag{A}$$

hold.

For the proof we fix the normal vectors e_ξ of the frame and perform the following transformation of the tangent vectors of the frame:

$$E_j = \frac{1}{2} e_j + \frac{1}{2} e_{n+j}, \qquad \bar{E}_j = \frac{1}{2i} e_j - \frac{1}{2i} e_{n+j} \qquad (j = 1, \dots, n).$$

In the transformed frame $(E_i, \bar{E}_i, e_\xi)$, the equalities (1) take the form

$$\Lambda^{\xi}_{ij} = 0, \qquad \Lambda^{\xi}_{\bar{i}\bar{j}} = 0, \qquad g_{ij} = g_{\bar{i}\bar{j}} = 0. \tag{1'}$$

From $(1')$ the fulfillment of condition (A) follows immediately.

Remarks. 1. Condition (A) contains the equality $\Lambda^{p\xi}_{p} = 0$. This means that strongly minimal surfaces are minimal.

2. The tensors $A^{\xi_1 \cdots \xi_{2k-1}}$ are tensors of mean curvatures in odd-dimensional directions.

3. Condition (A) is sufficient in the cases $n = 1$, $N - 2n = 1$. All two-dimensional minimal surfaces are strongly minimal.

3°. We pass to the clarification of the connections of strongly minimal surfaces with complex-analytic surfaces.

A **Kähler manifold** is a complex-analytic manifold on which a metric tensor is given satisfying the conditions $g_{J\bar{K}} = \bar{g}_{KJ}$, $Dg_{J\bar{K}}[dz^J d\bar{z}^K] = 0$ $(J = 1, \dots, N)$, where z^J are local coordinates on the manifold.

We attach the Kähler manifold to a moving frame, the components of whose infinitesimal displacement are complex linear differential forms ω^J, ω^J_K. The structural equations hold:

$$D\omega^J = [\omega^K \omega^J_K], \qquad D\omega^J_K = [\omega^L_K \omega^J_L] + R^J_{KL\bar{M}}[\omega^L \omega^{\bar{M}}],$$

$$dg_{J\bar{K}} = g_{J\bar{L}}\omega^{\bar{L}}_{\bar{K}} + g_{L\bar{K}}\omega^L_J.$$

We shall denote $\omega^{\bar{J}} = \bar{\omega}^J$, $\omega^{\bar{J}}_{\bar{K}} = \bar{\omega}^J_K$.

If one writes also the complex-conjugate equations, then the system obtained may be regarded as the structural equations of a $2N$-dimensional Riemannian manifold written in complex-conjugate coordinates. In this case the motions of the frame determined by the forms ω^P, ω^P_Q $(P = 1, \dots, N, \bar{1}, \dots, \bar{N})$ are restricted by the equalities

$$\omega^{\bar{J}}_K = 0, \qquad \omega^J_{\bar{K}} = 0.$$

If we free ourselves from this restriction, then we obtain a Riemannian manifold which we shall call a **Kähler manifold deprived of complex structure**.

A surface of a Kähler manifold, generally speaking not complex-analytic, may locally be given by the differential equations

$$\omega^J = \Lambda^J_k \pi^k + \Lambda^J_{\bar{k}} \pi^{\bar{k}} \quad (k = 1, \dots, n;\ \bar{k} = \bar{1}, \dots, \bar{n}),$$

where π^k, $\pi^{\bar{k}} = \overline{\pi^k}$ are invariant forms of the complex-analytic transformation group of the parameters.

The quantities $\Lambda^J_{\bar{k}}$ form the components of a differential-geometric object in the Kähler manifold, which we shall call the **object of analyticity**. The equality $\Lambda^J_{\bar{k}} = 0$ characterizes complex-analytic surfaces.

Theorem 2. *Surfaces of a Kähler manifold along which the object of analyticity is covariantly constant are strongly minimal surfaces in the Kähler manifold deprived of complex structure.*

Corollary. Complex-analytic surfaces of a Kähler manifold are strongly minimal in the Kähler manifold deprived of complex structure.

The proof of the theorem is based on the following assertion: if on a surface of a Riemannian space there exists a frame of zero order in which the second fundamental object and the metric tensor of the surface satisfy the relations

$$\Lambda^p_{ij} = -\Lambda^p_{n+i\, n+j}, \qquad \Lambda^p_{i\, n+j} = \Lambda^p_{n+i\, j}, \qquad g_{ij} = g_{n+i\, n+j}, \qquad g_{i\, n+j} = -g_{n+i\, j},$$

then such a surface is strongly minimal.

Remark. Since here everywhere only the local structure of a Kähler manifold is in question, in all arguments it may be ···

replace by Shirokov' s A-space ([5]) or, equivalently, by a pseudoholomorphic space (see, for example, ([6])).

4°. Let an N-dimensional unitary space U_N be given, i.e., a complex linear vector space with a real scalar product, and let E_J be a basis in U_N. By the **real plane** of the space U_N for the given basis we shall mean the set of real linear combinations of the vectors E_J; denote it by $R_N(E_J)$. The mapping

$$z^J E_J \to \frac{z^J + \bar{z}^J}{2} E_J$$

will be called the **projection** of the vector $z^J E_J$ onto $R_N(E_J)$.

A complex-analytic surface of a unitary space is given by the equation $z^J = F^J(p^i)$ $(i = 1, \dots, n)$, where F^J are analytic functions of the complex parameters p^i. In what follows we shall everywhere assume that $2n < N$.

Theorem 3. *A complex-analytic surface of a unitary space is projected onto the real plane $R_N(E_J)$ as a strongly minimal surface for some basis E_J.*

Analogously to projection onto $R_N(E_J)$, one can construct a projection onto the imaginary plane $I_N(E_J) = R_N(iE_J)$. The projections of an analytic surface onto $R_N(E_J)$ and $I_N(E_J)$ will be called **conjugate strongly minimal surfaces**. Mapping $I_N(E_J)$ onto $R_N(E_J)$ by the formulas $y^J(iE_J) \to y^J E_J$, we obtain in $R_N(E_J)$ a pair of conjugate strongly minimal surfaces. These surfaces are superposable; at corresponding points their tangent planes are parallel, as are their osculating planes of any order. Here there is a complete analogy with conjugate two-dimensional minimal surfaces in three-dimensional Euclidean space.

5°. We shall regard Euclidean space $\mathcal{E}_N$ as the real plane $R_N(E_J)$ in the unitary space U_N.

Theorem 4. *If in $\mathcal{E}_N = R_N(E_J)$ a strongly minimal surface S^* is given, then in U_N, with an arbitrary defined by constants, an analytic surface S can be constructed which is projected onto S^*.*

For minimal surfaces in $\mathcal{E}_3$, this assertion is given by the Schwarz formulas.

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Note: Figure translations are in progress. See original paper for figures.