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Abstract

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APPLICATION OF CHAPLYGIN' S METHOD TO THE SOLUTION OF THE CHARACTERISTIC CAUCHY PROBLEM FOR A SECOND-ORDER PARTIAL DIFFERENTIAL EQUATION OF HYPERBOLIC TYPE

(Presented by Academician S. L. Sobolev, 22 IX 1956)

In paper (1) it was established that for the equation

$$u_{xy} = f(x, y, u, p, q) \tag{1}$$

under the initial condition $u|_l = u(t),\ p|_l = p(t),\ q|_l = q(t)$, given along a curve l, where l is not a characteristic of equation (1), there holds a theorem analogous to Chaplygin' s theorem on differential inequalities (2).

In the present note we show the validity of the indicated theorem for equation (1) in the case where the value $u(x, y)$ is prescribed along the characteristics $x = x_0,\ y = y_0$.

Theorem. Let in equation (1) the function $f(x, y, u, p, q)$ be continuous, have continuous first-order partial derivatives with respect to all variables, and positive derivatives $f_u > 0,\ f_p > 0,\ f_q > 0$ in the domain

$$\overline{I}(x_0 \le x \le x_0 + \alpha, \quad y_0 \le y \le y_0 + \beta,\ \alpha > 0, \quad \beta > 0).$$

Let, in the indicated domain, $u(x, y)$ be a solution of equation (1) under the condition

$$u|_{x=x_0} = \varphi(y), \qquad u|_{y=y_0} = \psi(x), \qquad \varphi(y_0) = \psi(x_0), \tag{2}$$

where the functions $\varphi(y),\ \psi(x)$ are continuous together with their derivatives $\varphi'(y),\ \psi'(x)$, respectively for $y_0 \le y \le y_0 + \beta,\ x_0 \le x \le x_0 + \alpha$. Further, let $z(x, y)$ be a continuous function, have continuous second-order derivatives, satisfy conditions (2), and be a solution of the equation

$$z_{xy} = f(x, y, u, p, q) + \gamma(x, y), \tag{3}$$

where $\gamma(x, y) < 0$ in the domain $\overline{I}$.

Then in the domain $\overline{I}$

$$u(x, y) \geq z(x, y), \qquad u_x(x, y) \geq z_x(x, y), \qquad u_y(x, y) \geq z_y(x, y),$$

where equality will occur only on the half-lines $x = x_0,\ y \geq y_0;\ y = y_0,\ x \geq x_0$.

We outline the proof of the theorem.

From condition (2) and Chaplygin's theorem for an ordinary differential equation it follows that for $x = x_0,\ y > y_0$ the relations hold:

$$u(x_0, y) = z(x_0, y), \qquad u_y(x_0, y) = z_y(x_0, y), \qquad u_x(x_0, y) > z_x(x_0, y).$$

We shall show that in a neighborhood of the half-line $x = x_0,\ y > y_0$ the following inequalities are valid:

$$1)\ u_x(x, y) > z_x(x, y); \qquad 2)\ u(x, y) > z(x, y); \qquad 3)\ u_y(x, y) > z_y(x, y).$$

Indeed, inequality 1) follows from the inequality $u_x(x_0, y) > z_x(x_0, y)$ and from the continuity of the derivatives $u_x(x, y)$, $z_x(x, y)$.

Fixing $y = y'$, by the finite-increment formula we obtain $u(x, y') - z(x, y') = (x - x_0)[u_x(\xi, y') - z_x(\xi, y')]$, where $x_0 < \xi < x$, whence we conclude that $u(x, y) > z(x, y)$.

Inequality 3) follows from the equation

$$\frac{d}{dx}\left[u_y(x, y') - z_y(x, y')\right] = \left[u_y(x, y') - z_y(x, y')\right]\frac{\partial \bar{f}}{\partial q} + F(x, y'),$$

where $\partial \bar{f}/\partial q$ denotes the derivative $\partial f/\partial q$ at $x = \xi$, $y = y'$, while $F(x, y') > 0$ in that part of the domain I in which inequalities 1) and 2) hold.

The validity of inequalities 1), 2), and 3) near the half-line $y = y_0,\ x' > x_0$ is proved analogously.

Inequalities 1), 2), and 3) hold throughout the whole domain I. Indeed, taking into account the monotonicity of $f(x, y, u, p, q)$ with respect to the variables u, p, q and the continuity of $u(x, y)$, $z(x, y)$ and of their derivatives with respect to x, y, it is not difficult to show that on any subset $G \in I$ none of the equalities $u(x, y) = z(x, y)$, $u_x(x, y) = z_x(x, y)$, $u_y(x, y) = z_y(x, y)$ can occur.

This completes the proof of the theorem.

The construction of successive approximations to the solution of equation (1) under condition 2) can be carried out as shown in the work (1).

Kryvyi Rih
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CITED LITERATURE

[1] G. A. Artemov, DAN, **102**, No. 2 (1955).
[2] S. A. Chaplygin, *A New Method of Approximate Integration of Differential Equations*, 1950.

Note: Figure translations are in progress. See original paper for figures.