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R. V. GAMKRELIDZE

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Abstract

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R. V. GAMKRELIDZE

ON THE THEORY OF OPTIMAL PROCESSES IN LINEAR SYSTEMS*

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On the basis of the methods developed in ([1]), this note solves the problem of finding optimal processes in linear systems with one control parameter.

1°. Statement of the problem (see also ([1])); notation. A linear differential vector equation with one controlling (scalar) parameter u is given:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}u. \tag{1}$$

Here $\mathbf{x}$ is the representative point (vector) in the n-dimensional phase space X; $\mathbf{b}$ is a fixed vector of this space, A is a linear transformation of the space X (independent of time). The control function u is chosen in the class of piecewise-continuous functions (with a finite number of discontinuity points), not exceeding 1 in absolute value: $|u| \le 1$; such controls will be called admissible.

In the phase space X two points $\vec{\xi}_0, \vec{\xi}_1$ are given; it is required to choose such an admissible control $u = u(t)$ that the representative point $\mathbf{x}(t)$, moving along a trajectory of equation (1), passes from the position $\vec{\xi}_0$ to the position $\vec{\xi}_1$ in minimal time.

Such a control will be called an **optimal control**, and the corresponding trajectory an **optimal trajectory**.

Let $\vec{\varphi}_1(t), \dots, \vec{\varphi}_n(t)$ be contravariant vector functions with values in X, forming a fundamental system of solutions of the equation $\dot{\mathbf{x}} = A\mathbf{x}$. By $\vec{\psi}^1(t), \dots, \vec{\psi}^n(t)$ denote the covariant vector functions dual, respectively, to the functions $\vec{\varphi}_i(t)$: $\vec{\varphi}_i(t) \cdot \vec{\psi}^j(t) = \delta_i^j$. We have:

$$\dot{\vec{\varphi}}_i(t) = A\vec{\varphi}_i(t), \quad \dot{\vec{\psi}}^i(t) = -A'\vec{\psi}^i(t), \tag{2}$$

where A' is the linear transformation adjoint to A. Introduce n functions $h^i(t) = \vec{\psi}^i(t) \cdot \mathbf{b}$, $i = 1, \dots, n$. The solution $\mathbf{x}(t)$ of equation (1) with initial condition $\mathbf{x}(0) = \vec{\xi} = \vec{\varphi}_\alpha(0)\xi^\alpha$ is written in the form

$$\mathbf{x}(t) = \vec{\varphi}_\alpha(t) \left(\xi^\alpha + \int_0^t h^\alpha(\tau) u(\tau)\, d\tau \right).$$

2°. Nondegeneracy condition. We shall call equation (1) **nondegenerate** if the vector $\mathbf{b}$ does not lie in any invariant subspace of dimension $\leq n-1$ of the transformation A.

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* The results set forth in the present note were obtained in L. S. Pontryagin' s seminar on mathematical problems of the theory of oscillations and automatic regulation.

If equation (1) is degenerate, then either the transition time from $\vec{\xi}_0$ to $\vec{\xi}_1$ does not depend on the choice of the control function u, or the problem reduces to an analogous problem for an equation of lower order.

In what follows it is assumed that equation (1) is nondegenerate. In this case the vectors $\mathbf{b}, A\mathbf{b}, \ldots, A^{n-1}\mathbf{b}$ are independent, and, consequently, the functions $h^1(t), \ldots, h^n(t)$ are linearly independent.

3°. Equations for optimal controls and optimal trajectories. Let $u(t)$ be an optimal control; let $\mathbf{x}(t)$ be the corresponding optimal trajectory, joining the points $\vec{\xi}_0$ and $\vec{\xi}_1$; $\mathbf{x}(0) = \vec{\xi}_0$, $\mathbf{x}(t_1) = \vec{\xi}_1$. Then through every point of the trajectory $\mathbf{x}(t)$, $0 \leqslant t \leqslant t_1$, one can draw an $(n-1)$-dimensional hyperplane satisfying the following condition.

Denote by $\vec{\psi}(t)$ the covariant vector orthogonal to the hyperplane drawn through the point $\mathbf{x}(t)$ of the trajectory and uniquely determining this hyperplane. It turns out that for any admissible control $u(t) + \delta u(t)$ and the corresponding trajectory $\mathbf{x}(t) + \delta\mathbf{x}(t)$, where $\delta\mathbf{x}(0) = 0$, the inequality

$$\vec{\psi}(t) \cdot \delta\mathbf{x}(t) \leqslant 0$$

holds, and the vector function $\vec{\psi}(t)$ can be chosen so that it will satisfy the differential equation

$$\dot{\vec{\psi}} = -A'\vec{\psi}. \tag{3}$$

Consequently, $\vec{\psi}(t) = c_\alpha \vec{\psi}^\alpha(t)$, and we have:

$$\vec{\psi}(t) \cdot \delta\mathbf{x}(t) = c_\alpha \vec{\psi}^\alpha(t) \cdot \vec{\varphi}_\beta(t) \int_0^t h^\beta \delta u\, d\tau = \int_0^t \vec{\psi} \cdot \mathbf{b}\, \delta u\, d\tau \leqslant 0.$$

Since the functions $h^\alpha(t)$, $\alpha = 1, \ldots, n$, are linearly independent, $\vec{\psi}(t) \cdot \mathbf{b}$ is a nonzero solution of a linear differential equation of order n; and from the fact that $\delta u(t)$ is an arbitrary admissible variation of the optimal control $u(t)$, there follows the equality

$$u(t) = \text{sign}\ \vec{\psi}(t) \cdot \mathbf{b}. \tag{4}$$

In addition to the conditions listed, one more is satisfied:

$$\vec{\psi}(t) \cdot \dot{\mathbf{x}}(t) = \vec{\psi}(t) \cdot [A\mathbf{x}(t) + \mathbf{b}\, u(t)] = \text{const} \geqslant 0. \tag{5}$$

Combining equations (1), (3)–(5), we obtain the theorem:

All optimal controls $u(t)$ and the corresponding optimal trajectories $\mathbf{x}(t)$, issuing at $t = 0$ from the point $\vec{\xi}$, are contained among the controls u and the corresponding trajectories obtained by solving the system of equations

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}\, u, \quad \mathbf{x}(0) = \vec{\xi}; \quad \dot{\vec{\psi}} = -A'\vec{\psi}, \quad u = \text{sign}\ \vec{\psi} \cdot \mathbf{b}. \tag{6}$$

The initial value $\vec{\psi}(0)$ of the solution $\vec{\psi}(t)$ is subject to the sole condition

$$\vec{\psi}(0) \cdot [A\mathbf{x}(0) + \mathbf{b}\, u(0)] \geqslant 0.$$

The system of equations (6) exactly expresses the maximum principle formulated in note (1).

4°. The problem of synthesis of an optimal system. In the theory of automatic control one is interested in the optimal transition

along the trajectory of equation (1) from an arbitrary initial position x to the origin. The set M of those points x of the phase space X from which one can reach the origin by an admissible control and, consequently, by an optimal control, is open and convex. If the transformation A has stable eigenvalues, then the set M coincides with the whole space X.

From any point x, no more than one trajectory of equation (1), satisfying the maximum principle (6), leads to the origin.

Therefore, a real function $u(x)$ of the vector argument x is uniquely defined, possessing the property that, if one moves along the trajectories of the equation

$$\dot{x} = Ax + bu(x),$$

then one reaches the origin in the minimal time from any prescribed initial position, provided that from it one can reach the origin at all by some admissible control.

The computation of the function $u(x)$ is called the **synthesis of the optimal system** (1). This computation can be carried out on the basis of formulas (6). Having prescribed an arbitrary initial value $\vec{\psi}(0)$ for $\vec{\psi}$ and the initial value $x(0) = 0$, one must solve system (6) on the half-axis $-\infty < t \leq 0$. Since the function $u(x)$ assumes three values: $1, 0, -1$, in order to determine it it suffices to know the set of points of "switching" of the control $u(x)$, i.e. the set of values x satisfying the equation $u(x) = 0$, as well as those regions into which this set divides the space X.

In the case of a second-order equation, in the phase plane one obtains switching lines; finding them on the basis of system (6) is a completely elementary problem. These lines were first obtained by Bushaw (see ([2])).

If the transformation A has real eigenvalues, then the set of switching points of the control function $u(x)$ is a hypersurface. A method for constructing this hypersurface was first indicated by A. A. Feldbaum ([3]). On the basis of equations (6) it is easy to obtain a parametric representation of this hypersurface. Let $t_1, t_2, \dots, t_{n-1}$ be parameters subject to the single condition $0 \geq t_1 \geq t_2 \geq \dots \geq t_{n-1}$; then the parametric representation of the switching hypersurface will be

$$x(t_1, \dots, t_{n-1}) =$$

$$= \pm\vec{\varphi}_\alpha(t_{n-1}) \left[\int_0^{t_1} \vec{\psi}^\alpha \cdot b \, d\tau - \int_{t_1}^{t_2} \vec{\psi}^\alpha \cdot b \, d\tau + \dots + (-1)^{n-2} \int_{t_{n-2}}^{t_{n-1}} \vec{\psi}^\alpha \cdot b \, d\tau \right]. \tag{7}$$

This hypersurface divides the space into two connected regions; in one of them the function assumes the value $+1$, in the other the value -1.

In the general case of complex roots, the set of switching points of the control function u is a pseudomanifold, and its parametric representation, analogous to (7), cannot be obtained. However, it can be computed on the basis of equations (6) with arbitrarily great accuracy.

Steklov Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.