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Abstract

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ON THE COMBINATORIAL INVARIANCE OF PONTRYAGIN CLASSES

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1. Formulation of the main results. In this paper we prove the combinatorial invariance of the reduced Pontryagin classes p_{4k}, i.e., of the characteristic Pontryagin classes considered as classes of weak ∇-homology. The paper is closely connected with the work (1), to which we shall refer repeatedly. For the definition of a C^1-triangulation see (2). If φ is a continuous mapping, then φ^* and φ_* denote the corresponding homomorphisms of the groups of ∇- and Δ-homology.

Main theorem. *Let M_0^n, M_1^n be smooth closed manifolds possessing isomorphic C^1-triangulations K_0, K_1, and let $\varphi : M_1^n \to M_0^n$ be a homeomorphic mapping determined by some isomorphism between K_0 and K_1. Then*

$$\varphi^*(p_{4k}(M_1^n)) = p_{4k}(M_0^n) \quad (k = 1, 2, ...).$$

As a consequence we obtain:

If smooth closed orientable manifolds M_0^{4l}, M_1^{4l} possess isomorphic C^1-triangulations, then, with a suitable orientation, they have the same Pontryagin numbers.

As is known, two simplicial complexes are called **combinatorially equivalent** if they have isomorphic subdivisions. We shall call two triangulations of one and the same polyhedron **strictly equivalent** if there exist isomorphic subdivisions of them and such an isomorphism between these subdivisions that the corresponding homeomorphism of the polyhedron is homotopic to the identity. For example, as follows from Whitehead' s results (2), any two C^1-triangulations of a smooth closed manifold are strictly equivalent. According to the so-called "main hypothesis of combinatorial topology," any two triangulations of one and the same polyhedron are combinatorially equivalent. This hypothesis becomes still stronger if combinatorial equivalence is replaced by strict equivalence. From our main theorem it immediately follows:

If the main hypothesis of combinatorial topology is true, then the Pontryagin numbers are topological invariants. If the strengthened main hypothesis is true, then the reduced Pontryagin classes are topological invariants.

Since the combinatorial invariance of the classes p_{4k} has been established, the question arises of defining them for combinatorial manifolds. We give such a definition for the classes σ_{4k} (^{1}). The classes p_{4k} can be defined in terms of the classes σ_{4k}.

The further results are obtained by comparing what was set out above with the recent work of Milnor (^{3}). They concern the problem of introducing smoothness into a combinatorial manifold and are presented in § 5. We note that in Milnor' s work a hypothesis is refuted which in the theory of smooth manifolds served as an analogue of the main hypothesis of combinatorial topology: if two smooth manifolds are homeomorphic, then they are smoothly homeomorphic. It is not hard to show that homeomorphic but not smoothly homeomorphic mani-

Milnor manifolds M_k^7 have isomorphic C^1-triangulations. Thus, under the conditions of the main theorem the manifolds M_0^1 and M_1^n need not be smoothly homeomorphic.

2. Lemmas. In what follows, by Σ^r we mean a sphere triangulated by projecting onto it from its center the boundary of a regular $(r+1)$-dimensional simplex inscribed in it, and by c the center of one of its r-dimensional simplices.

Lemma 1. *Let M^n be a smooth closed manifold and $f : M^n \to \Sigma^r$ a smooth map. If c is a regular point with respect to f (i.e., at all points of the set $f^{-1}(c)$ the functional matrix of the map f has rank r), then there exists a C^1-triangulation of the manifold M^n and a map $\bar{f} : M^n \to \Sigma^r$, simplicial with respect to it and homotopic to f, such that $\bar{f}^{-1}(c) = f^{-1}(c)$.*

Proof. Introduce a Riemannian metric in M^n. For sufficiently small ε, the differential properties of the map f determine a natural decomposition of the ε-neighborhood of the submanifold $N^{n-r} = f^{-1}(c)$ into the direct product $N^{n-r} \times V^r$ of this submanifold by an r-dimensional ball, where the fiber $N^{n-r} \times b$, corresponding to the center b of the ball, coincides with N^{n-r}, and the fibers $a \times V^r$ $(b \in N^{n-r})$ are geodesic balls of radius ε, normal to N^{n-r} (^{4}). Let T^r be a simplex, interior to V^r, with center at the point b and with vertices $b_0, \dots, b_r$, and let K be some C^1-triangulation of the product $N^{n-r} \times T^r \subset N^{n-r} \times V^r$, having the property that all its vertices lie in the layers $N^{n-r} \times b_j$. Such a triangulation exists: it suffices to take any C^1-triangulation K' of the manifold N^{n-r}, enumerate its vertices in the sequence $a_0, \dots, a_q$, take as vertices in K the points $a_i \times b_j$, and declare $(a_{i_0} \times b_{j_0}, a_{i_1} \times b_{j_1}, \dots, a_{i_\alpha} \times b_{j_\alpha})$, where $i_0 \leq i_1 \leq \dots \leq i_\alpha$, to be a simplex in K, if $(a_{i_0}, a_{i_1}, \dots, a_{i_\alpha})$ is a simplex in K' and $j_0 \leq j_1 \leq \dots \leq j_\alpha$. We extend the triangulation K to all of M^n; the existence of such an extension, as well as the existence of the original triangulation K', follows from the results of Whitehead (^{2}).

Let $c_0, \dots, c_r, c_{r+1}$ be the vertices in Σ^r, with the vertex opposite c being c_{r+1}. Construct the simplicial map $\bar{f} : M^n \to \Sigma^r$ by setting $\bar{f}(a_i \times b_j) = c_j$ $(i = 0, \dots, q;\ j = 0, \dots, r)$ and $\bar{f}(x) = c_{r+1}$ for every other vertex $x \in M^n$. It is not difficult to verify that $\bar{f}^{-1}(c) = N^{n-r}$ and that $\bar{f}$ is homotopic to f.

Lemma 2. *Let M be a polyhedron; L_0 and L_1 its triangulations; g_0 and g_1 its maps into a triangulated polyhedron Σ, simplicial with respect to L_0 and L_1 and homotopic to one another; and let $M \times I$ be its product with the interval $[0,1]$. If L_0 and L_1 have a common subdivision, then there exist a triangulation L of the prism $M \times I$ and a map $g : M \times I \to \Sigma$, simplicial with respect to L, such that on $M \times 0$ they coincide with $L_0 \times 0$ and $g_0 \times 0$, and on $M \times 1$ with $L_1 \times 1$ and $g_1 \times 1$.*

The proof is elementary.

3. Proof of the main theorem. We may assume M_0^n and M_1^n to be orientable, since the nonorientable case reduces to the orientable one by passing to the orientable two-sheeted covering (cf. (^{1})). Next, it is possible to restrict ourselves to the case of odd n, to which the case of even n reduces by multiplying the manifolds M_0^n and M_1^n by a circle. Finally, instead of the system of relations $\varphi^*(p_{4k}(M_1^n)) = p_{4k}(M_0^n)$ $(k = 1, 2, ...)$, it suffices to prove the equivalent system of relations $\varphi^*(s_{4k}(M_1^n)) = s_{4k}(M_0^n)$ $(k = 1, 2, ...)$ (^{1}).

Orient Σ^r and M_0^n, and denote by $H_{4k} = H_{4k}(M_0^n)$ the integral $4k$-dimensional Δ-homology group of the manifold M_0^n. By the theorem of Serre (^{5})—Thom (^{4}), there exists a natural number m (depending on n and k) such that every class of the form mu', $u' \in H_{4k}$, contains an oriented submanifold N_0^{4k} of the smooth manifold M_0^n, serving as the oriented inverse image of a point $c \in \Sigma^r$, $r = n-4k$, under some smooth map $f_0 : M_0^n \to \Sigma^r$, for which c is a regular point: $N_0^{4k} = f_0^{-1}(c)$. Transfer the orientation of M_0^n by means of φ to M_1^n, construct for the map $f_0\varphi^{-1} : M_1^n \to \Sigma^r$ a smooth map $f_1 : M_1^n \to \Sigma^r$ homotopic to it, for which c is also a regular point, and put $N_1^{4k} = f^{-1}(c)$. The submanifolds N_0^{4k} and N_1^{4k} are normalizable (i.e., possess a system of $n-4k$ independent exterior vector fields) and belong to the classes $u = mu'$ and $\varphi_* u$, respectively, whence $(s_{4k}(M_0^n), u) = \alpha_k \sigma(N_0^{4k})$ and $(s_{4k}(M_1^n), \varphi_* u) = \alpha_k \sigma(N_1^{4k})$ (^{1}). Therefore, to prove the relations $\varphi^*(s_{4k}(M_1^n)) = s_{4k}(M_0^n)$ it is enough to show that $\sigma(N_1^{4k}) = \sigma(N_0^{4k})$.

Apply Lemma 1 to $f_i : M_i^n \to \Sigma^r$ $(i = 0, 1)$: let $\overline{K}_i$ be the triangulation and $\overline{f}_i$ the map whose existence it asserts. On M_i^n we have two C^1-triangulations K_i and $\overline{K}_i$, and, according to Whitehead' s results (see § 1), there exist isomorphic subdivisions K_i', $\overline{K}_i'$ of the complexes K_i, $\overline{K}_i$ and an isomorphism $K_i' \to \overline{K}_i'$ between these subdivisions such that the corresponding homeomorphism $\varphi_i : M_i^n \to M_i^n$ is homotopic to the identity. Consider the homeomorphism $\psi : M_0^n \to M_1^n$ defined by the formula $\psi = \varphi_1 \varphi \varphi_0^{-1}$, and, bearing in mind the notation of Lemma 2, set $M = M_1^n$, $L_0 = \psi(\overline{K}_0)$, $L_1 = \overline{K}_1$, $\Sigma = \Sigma^r$, $g_0 = \overline{f}_0 \psi^{-1}$, $g_1 = \overline{f}_1$. It is not difficult to verify that g_0 and g_1 are homotopic to one another. Further, $\varphi(K_0')$ and K_1', as subdivisions of one and the same complex K_1, have a common subdivision K, and it is clear that $\varphi_1(K)$ is a common subdivision of the complexes L_0 and L_1. Thus Lemma 2 is applicable. Orient the prism $M \times I$ so that $\Delta(M \times I) = M \times 1 - M \times 0$, and set (in the notation of Lemma 2) $N = g^{-1}(c)$. This is an oriented $(4k+1)$-dimensional h-manifold with boundary (its orientation is determined by the orientations of Σ^r and $M \times I$ and by the map g), and $\Delta N = N_1 - N_0$, where $N_0 = \psi(N_0^{4k}) \times 0$, $N_1 = N_1^{4k} \times 1$. Consequently, $\sigma(N_1) = \sigma(N_0)$ (6,7) and $\sigma(N_1^{4k}) = \sigma(N_0^{4k})$.

Remark. The main theorem is also true for open manifolds and manifolds with boundary. The proof becomes somewhat more complicated.

4. Definition of the classes σ_{4k} for combinatorial manifolds. This definition is suitable for any combinatorial h-manifold, i.e., a complex K^n whose body $|K^n|$ is an h-manifold (the presence of a boundary is also allowed). Suppose first that n is odd and $|K^n|$ is oriented and closed, and orient $|K^n|$ and Σ^r, $r = n - 4k$. From the theorem of Serre (5) it is not difficult to derive that there exists a natural number m (depending only on n and k) such that in every class mu, $u \in H_{4k}(K^n)$, there is an oriented h-manifold of the form $f^{-1}(c)$, where f is a map of the polyhedron $|K^n|$ into Σ^r, simplicial with respect to some subdivision K_1 of the complex K^n. It turns out that the number τ/m, where τ is the signature of this h-manifold, is uniquely determined by the class u (i.e., does not depend on the choice of m, K_1, and f) and is a linear function of u; thus the formula $(\sigma_{4k}, u) = \tau/m$ defines a certain class $\sigma_{4k} = \sigma_{4k}(K^n)$ of rational ∇-homology of the polyhedron $|K^n|$. If n is even, then $\sigma_{4k}(K^n)$ is defined as $i^*(\sigma_{4k}(K^n \times S^1))$, where S^1 is a triangulation of the circle; i is the natural embedding $K^n \to K^n \times S^1$. If

If $|K^n|$ is nonorientable, then $\sigma_{4k}(K^n)$ is defined by passing to the orientable double covering. If K^n has a boundary, then $\sigma_{4k}(K^n)$ is defined by doubling K^n.

This definition is combinatorially invariant. σ_{4k} is a rational V-homology class that becomes integral after multiplication by a number depending only on n and k. If K^n is a C^1-triangulation of a smooth manifold, then such a number is α_k, namely $\alpha_k \sigma_{4k} = s_{4k}$. In the general case this formula may be taken as the definition of the classes s_{4k}, and hence of the classes p_{4k}.

The proof of the assertions made essentially repeats the arguments of § 3. However, it also relies on Lemma 3, in which, as above, $|K^n|$ is closed, K^n and Σ^r are oriented, n is odd, and $r = n - 4k$.

Lemma 3. Let $f_0 : |K^n| \to \Sigma^r$ and $f_1 : |K^n| \to \Sigma^r$ be maps that are simplicial with respect to certain subdivisions of the complex K^n. If the oriented h-manifolds $f_0^{-1}(c)$ and $f_1^{-1}(c)$ belong to the same class of integral Δ-homologies, then there exists a map $g : \Sigma^r \to \Sigma^r$ such that gf_0 and gf_1 are homotopic.

The proof is analogous to the proof of Serre' s theorem (5).

5. **Combinatorial manifolds and smoothness.** We shall call a formal manifold (2) **admitting smoothness** if it is combinatorially equivalent to a C^1-triangulation of some smooth manifold. We shall call a closed oriented combinatorial h-manifold K^n **internally homologous to a smooth manifold** if there exists a smooth oriented manifold with a C^1-triangulation K_1^n such that the difference $K_1^n - K_0^n$ is isomorphic to the

> boundary of some oriented combinatorial h-manifold K^{n+1}. From Cairns' s results (^{8}) it follows that every formal manifold K^n with $n \leq 4$ admits smoothness.

Consider the manifolds B_k^8 and X_k^8, constructed by Milnor (^{3}). Let L_k^8 be a triangulation of the second, obtained from some C^1-triangulation of the first. L_k^8 is a formal manifold. We shall show that:

For $k^2 \not\equiv 1 \pmod 7$*, the closed oriented formal manifold* L_k^8 *admits no smoothness and, moreover, is not internally homologous to a smooth manifold.*

Proof. According to § 4, for L_k^8 the class $\frac{1}{7}(9\sigma_4^2 + 45\sigma_8)$ is defined, and from Milnor' s results it is not difficult to derive that this class cannot be integral. Meanwhile, for a smooth manifold it, being equal to p_8, is integral; and for a combinatorial h-manifold internally homologous to a smooth one, it therefore must also be integral.

As a consequence we obtain:

If the fundamental hypothesis of combinatorial topology is true, then no smoothness can be introduced on the topological manifold X_k^8*.*

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