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Abstract

Full Text

Mathematics

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A GENERALIZATION OF M. G. KREIN' S THEOREM

(Presented by Academician S. N. Bernstein on 27 IV 1957)

In this note we consider meromorphic functions of the form

$$f(z) = \sum \frac{A_k}{z - h_k}, \quad (1)$$

where: 1) $\sum |\operatorname{Im} h_k^{-1}| < \infty$, 2) $\sum |A_k| < \infty$.

M. G. Krein showed ¹ that for such a function, if one additionally assumes that it has no zeros,

$$T_f(r) < C_f(r + 1),$$

where C_f is some constant, and $T_f(r)$ is the Nevanlinna characteristic of the meromorphic function. Here we remove the assumption that there are no zeros and establish the following theorem.

Theorem 1. For a function of the form (1) the following asymptotic inequality holds*

$$T_f(r) \leq N_f(\theta r, 0) + C_{f,\theta} r, \quad (2)$$

in which θ is an arbitrary number greater than one; $C_{f,\theta}$ is some constant.

Proof. 1°. By Nevanlinna' s first theorem,

$$T_f(\rho) = N_f(\rho, 0) + m_f(\rho, 0) + O(1).$$

Multiplying both sides of this relation by ρ , integrating with respect to ρ from r to θr , and finally using the monotonicity of T_f and N_f , we obtain

$$T_f(r) \leq N_f(\theta r, 0) + \frac{2}{(\theta^2 - 1)^2 r^2} \int_r^{\theta r} m_f(\rho, 0) \rho d\rho + O(1). \quad (3)$$

We need to show that

$$\int_r^{\theta r} m_f(\rho, 0) \rho d\rho < C'_{f, \theta} r^3.$$

For this we shall use the inequality

$$\begin{aligned} \int_r^{\theta r} m_f(\rho, 0) \rho d\rho &= \int_{r \leq |z| \leq \theta r} \ln_- |f(z)| dx dy \leq (1 + \theta^2 r^2) \int_{|z| \leq \theta r} \frac{\ln_- |f(z)|}{1 + x^2 + y^2} dx dy \leq \\ &\leq (1 + \theta^2 r^2) \int_{-\theta r}^{\theta r} \int_{-\infty}^{\infty} \frac{\ln_- |f(z)|}{1 + x^2 + y^2} dx dy = (1 + \theta^2 r^2) \left\{ \int_{-1}^1 \int_{-\infty}^{\infty} + \int_1^{\theta r} \int_{-\infty}^{\infty} + \int_{-\theta r}^{-1} \int_{-\infty}^{\infty} \right\}. \end{aligned} \quad (4)$$

* This proposition was put forward as a hypothesis by B. Ya. Levin (oral communication).

Let us estimate the integrals in the brackets.

2°. Assuming $|h_k| > 1$ ($k = 0, \pm 1, \pm 2, \dots$) (which entails no loss of generality), we draw around each point h_k a circle (C_k) of radius $|\operatorname{Im} h_k|^{-1}$. For nonreal z lying outside the circles (C_k), we have

$$|f(z)| \leq \sum_{|z-h_k| < y/2} \frac{|A_k|}{|z-h_k|} + \sum_{|z-h_k| \geq y/2} \frac{|A_k|}{|z-h_k|} \leq \frac{C_1}{|y|} (|z|^2 + 1).$$

Putting $g(z) = f(z)\chi(\{h_k\}, z)$, where

$$\chi(\{h_k\}, z) = \prod_{\operatorname{Im} h_k > 0} \left(1 - \frac{z}{h_k}\right) \left(1 - \frac{z}{\bar{h}_k}\right)^{-1},$$

we obtain, by the maximum-modulus principle, the estimate for $g(z)$ when $\operatorname{Im} z > 0$:

$$|g(z)| \leq C_2 |y|^{-1} (|z|^2 + 1).$$

Hence the inequality follows

$$\int_{-\infty}^{\infty} \frac{\ln_+ |g(t + ih)|}{1 + t^2} dt < C_3 \ln \frac{1}{h} + C_4 \quad (0 < h \leq 1). \quad (5)$$

We shall next need the following fact (2). If $F(z)$ is holomorphic for $\text{Im } z > 0$, continuous for $\text{Im } z \geq 0$, and

$$1) \quad \frac{\ln |F(z)|}{|z|} \leq C_F < \infty,$$

and

$$2) \quad \int_{-\infty}^{\infty} \frac{\ln |F(z)|}{1+t^2} dt \quad \text{converges,}$$

then

$$\ln |F(z)| = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\ln |F(t)|}{(x-t)^2 + y^2} dt + \ln |\chi(\{b_k\}, z)| + k_F y, \quad (6)$$

where b_k are the zeros of $F(z)$, and k_F is a constant.

Applying (6) to $g(z + ih)$ and then putting $z = i$, we obtain

$$\begin{aligned} & \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln_- |g(t + ih)|}{1+t^2} dt = \\ & = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln_+ |g(t + ih)|}{1+t^2} dt + k_g + \ln |\chi(\{a_k - ih\}, z)| - \ln |g((h+1)i)|, \end{aligned}$$

where a_k are the zeros of $f(z)$. Hence, in view of (5), it follows that

$$\int_{-\infty}^{\infty} \frac{\ln_- |g(t + ih)|}{1+t^2} dt < C_5 \ln \frac{1}{h} + C_6 \quad (0 < h \leq 1),$$

and, since $|f(t + ih)| \geq |g(t + ih)|$ for $h > 0$, then

$$\int_{-\infty}^{\infty} \frac{\ln_- |f(t + ih)|}{1+t^2} dt < C_5 \ln \frac{1}{h} + C_6 \quad (0 < h \leq 1).$$

Integrating the last inequality with respect to h from 0 to 1, we shall have

$$\int_0^1 \int_{-\infty}^{\infty} \frac{\ln_- |f(t + ih)|}{1+t^2} dh dt < C_7.$$

Estimating in the same way the integral from -1 to 0, we obtain

$$\int_{-1}^1 \int_{-\infty}^{\infty} \frac{\ln_- |f(t+ih)|}{1+t^2} dh dt < C_8. \quad (7)$$

3°. Let us estimate the two other integrals in (4). Using (6), we have for $y > 1$

$$\begin{aligned} \ln_- |f(x+iy)| \leq \ln_- |g(x+iy)| &\leq \frac{y-\frac{1}{2}}{\pi} \int_{-\infty}^{\infty} \frac{\ln_- |g(t+i/2)|}{(x-t)^2 + (y-\frac{1}{2})^2} dt + \\ &+ |k|(y-\frac{1}{2}) + \ln_- |\chi(\{a_k - i/2\}, z - i/2)|. \end{aligned}$$

Dividing this inequality by $1+x^2+(y-\frac{1}{2})^2$ and integrating with respect to x , we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\ln_- |f(x+iy)|}{1+x^2+(y-\frac{1}{2})^2} dx &\leq \frac{y-\frac{1}{2}}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\ln_- |g(t+i/2)|}{(x-t)^2 + (y-\frac{1}{2})^2} \frac{dx dt}{1+x^2+(y-\frac{1}{2})^2} + \\ &+ |k| + \int_{-\infty}^{\infty} \frac{\ln_- |\chi(\{a_k - i/2\}, z - i/2)|}{1+x^2+(y-\frac{1}{2})^2} dx = I_1 + |k| + I_2. \end{aligned} \quad (8)$$

Further, it is easy to see that

$$\begin{aligned} I_1 &= \frac{y-\frac{1}{2}}{\pi} \int_{-\infty}^{\infty} \ln_- |g(t+i/2)| \left\{ \int_{-\infty}^{\infty} \frac{dx}{[(x-t)^2 + (y-\frac{1}{2})^2][1+x^2+(y-\frac{1}{2})^2]} \right\} dt \leq \\ &\leq \frac{y-\frac{1}{2}}{\pi} \int_{-\infty}^{\infty} \ln_- |g(t+i/2)| \frac{2\pi}{(y-\frac{1}{2})(t^2+1)} dt < C_9. \end{aligned} \quad (9)$$

Introducing $a'_k = a_k - i/2$, $z' = z - i/2$, and denoting these quantities again by a_k and z , consider

$$I_2 = \int_{-\infty}^{\infty} \frac{\ln_- |\chi(\{a_k\}, z)|}{1+x^2+y^2} dx \quad \left(a_k = \alpha_k + i\beta_k, \beta_k > 0, \sum \frac{\beta_k}{\alpha_k^2 + \beta_k^2} < \infty \right).$$

From the identity

$$\int_{-\infty}^{\infty} \frac{\ln |(x-a)^2 + (y-b)^2|}{1+x^2+y^2} dx =$$

$$= \begin{cases} \frac{\pi}{\sqrt{1+y^2}} \ln \left[a^2 + (y-b + \sqrt{1+y^2})^2 \right], & (y-b > 0), \\ \frac{\pi}{\sqrt{1+y^2}} \ln \left[a^2 + (-y+b + \sqrt{1+y^2})^2 \right], & (y-b \leq 0), \end{cases}$$

it follows that

$$\begin{aligned} \frac{2\sqrt{1+y^2}}{\pi} I_2 &= \frac{\sqrt{1+y^2}}{\pi} \sum \int_{-\infty}^{\infty} \ln \frac{(\alpha_k - x)^2 + (\beta_k + y)^2}{(\alpha_k - x)^2 + (\beta_k - y)^2} \frac{dx}{1+x^2+y^2} = \\ &= \sum_{y < \beta_k} \ln \frac{\alpha_k^2 + (y + \beta_k + \sqrt{1+y^2})^2}{\alpha_k^2 + (-y + \beta_k + \sqrt{1+y^2})^2} + \sum_{y > \beta_k} \ln \frac{\alpha_k^2 + (y + \beta_k + \sqrt{1+y^2})^2}{\alpha_k^2 + (y - \beta_k + \sqrt{1+y^2})^2} \leq \\ &\leq 4y \sum_{y < \beta_k} \frac{\beta_k + \sqrt{1+y^2}}{\alpha_k^2 + (-y + \beta_k + \sqrt{1+y^2})^2} + \\ &+ 4(y + \sqrt{1+y^2}) \sum_{y < \beta_k} \frac{\beta_k}{\alpha_k^2 + (y - \beta_k + \sqrt{1+y^2})^2} \leq \\ &\leq 8y \sum \frac{\beta_k}{\alpha_k^2 + \beta_k^2} + 4(y + \sqrt{1+y^2}) \sum \frac{\beta_k}{\alpha_k^2 + \beta_k^2} \leq C_{10}(y+1), \end{aligned}$$

whence $I_2 \leq C_{11}$. Substituting the last inequality from (9) into (8), and then integrating from 1 to θr , we obtain

$$\int_1^{\theta r} \int_{-\infty}^{\infty} \frac{\ln_- |f(z)|}{1+x^2+y^2} dy dx \leq C_{12}r; \quad \int_{-\theta r}^{-1} \int_{-\infty}^{\infty} \frac{\ln_- |f(z)|}{1+x^2+y^2} dy dx \leq C_{13}r \tag{10}$$

(the validity of (10) is obtained by applying the arguments of this item to $f(z)$ for $\text{Im } z \leq -1$).

4°. Substituting (7) and (10) into (4), we obtain

$$\int_r^{\theta r} m_f(\rho, 0) \rho d\rho < C'_{f,\theta} r^3. \tag{11}$$

From this inequality, in view of (3), (2) follows. Theorem 1 is proved.

5°. Inequality (11) makes it possible to establish the following theorem:

Theorem 2. *If for a function of the form (1) $\lim_{r \rightarrow \infty} T_f(r)r^{-1} = \infty$, then for every a , including $a = \infty$, $\delta_f(a) = 0$.*

This proposition is adjacent to the following result of M. V. Keldysh ⁽³⁾:

A meromorphic function of finite order $f(z) = \sum \frac{A_k}{z - h_k}$ has no nonzero deficient values, and if $\sum |A_k| < \infty$ and $\sum A_k \neq 0$, then also $\delta_f(0) = 0$.

From Theorem 2 (and the remark, see below) it follows, in particular, that if $\sum |\operatorname{Im} h_k^{-1}| < \infty$, then for the function $f(z)$ the equality $\delta_f(0) = 0$ holds also without these additional assumptions.

Proof of Theorem 2. In view of (11) we have

$$\frac{C'_{\theta, f} r}{T_f(r)} \geq \frac{\int_r^{\theta r} m_f(\rho, 0) \rho d\rho}{r^2 T_f(r)} = \frac{m_f(\theta_r r, 0) \theta_r r}{r^2 T_f(r)} (\theta - 1) r > \frac{m_f(\theta_r r, 0)}{T_f(\theta_r r)} (\theta - 1),$$

where $1 < \theta_r < \theta$, whence

$$\delta_f(0) = \lim_{\rho \rightarrow \infty} \frac{m_f(\rho, 0)}{T_f(\rho)} = 0.$$

The equality $\delta_f(a) = 0$ for $a \neq 0, \infty$ follows from consideration of the function $f_1 = (f - a)z^{-1}$, which is represented in the form (1), and $m_{f_1}(r, 0) \geq m_f(r, a)$. The equality $\delta_f(\infty) = 0$ was proved by M. V. Keldysh ⁽³⁾ also for functions of infinite order.

Remark. The assertions of Theorems 1 and 2 remain valid if the condition $\sum |A_k| < \infty$ is dropped, for, if the latter is not fulfilled for $f(z)$, then, by virtue of the absolute convergence of the series (1), it is fulfilled for $f(z)z^{-1}$.

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Note: Figure translations are in progress. See original paper for figures.

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