MATHEMATICS

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Abstract

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MATHEMATICS

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ON THE PROBLEM OF ANALYTIC CONTINUATION OF THE PRINCIPAL SOLUTIONS OF AN EQUATION OF HYPERBOLIC TYPE WITH SINGULAR COEFFICIENTS

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Consider, in the half-plane $y > x$, the equation

$$(y-x)z_{xy} + \beta(z_x - z_y) + c(x,y)z = 0, \qquad c(x,y) \geq 0, \qquad 0 < \beta < \tfrac{1}{2}, \quad (1)$$

assuming that

$$c(x,y) = \sum_{k=0}^{\infty} c_{2k}(y-x)^{1+2k}; \qquad c_{2k} = \text{const}.$$

Let $\overline{D}$ be the closed domain bounded by the segment MN of the line $y = x$ and by the characteristics MP and NP of equation (1), issuing from the points $M(x_1, x_1)$ and $N(x_2, x_2)$. We shall call problem K for equation (1) the singular Cauchy problem

$$z(x,x) = \tau(x), \qquad z_\zeta(x,x) = \nu(x); \qquad \zeta = -\left(\frac{y-x}{2-2a}\right)^{1-a}, \qquad a = 2\beta, \quad (2)$$

and the problems T_1 and T_2 the Tricomi problems

$$z(x_1,y) = 0, \quad z_\zeta(x,x) = \nu(x); \qquad z(x_1,y) = 0, \quad z(x,x) = \tau(x), \quad \tau(x_1) = 0. \quad (3)$$

Here $\tau(x)$ and $\nu(x)$ are twice continuously differentiable functions on the interval $x_1 \leq x \leq x_2$. Using as majorants the principal solutions constructed earlier ([1,2]) in explicit form for the case $4c(x,y) = b^2(y-x)$, $b = \text{const}$, one can prove the following theorems.

Theorem 1. There exist unique solutions of the problems K, T_1, and T_2, twice continuously differentiable in the domain D. These solutions depend continuously on the initial functions $\tau(x)$ and $\nu(x)$, and, moreover, for each of the problems K, T_1, and T_2 zero-order correctness holds ([3]).

Theorem 2. The solution z_0 of problem K can be represented in the form

$$z_0 = \gamma_1 (y-x)^{1-a} \int_x^y \tau(x')[(x'-x)(y-x')]^{\beta-1} R_{\beta-1}(x'-x, y-x')\, dx'$$

$$- \gamma_2 \int_x^y \nu(x')[(x'-x)(y-x')]^{-\beta} R_{-\beta}(x'-x, y-x')\, dx', \tag{4}$$

while the problems T_1 and T_2 have solutions z_1 and z_2 of the form

$$z_1 = \gamma \int_{x_1}^x \nu(x')[(x-x')(y-x')]^{-\beta} \overline{R}_{-\beta}(x-x', y-x')\, dx', \tag{5a}$$

$$z_2 = k(y-x)^{1-a} \int_{x_1}^x \tau(x')[(x-x')(y-x')]^{\beta-1} \overline{R}_{\beta-1}(x-x', y-x')\, dx'. \tag{5b}$$

Here $\gamma_1 = \Gamma(a)/\Gamma^2(\beta)$; $\gamma_2 = \Gamma(2-a)/\Gamma^2(1-\beta)$; $k = \Gamma(1-\beta)/\Gamma(\beta)\Gamma(1-a)$; $\gamma = k\gamma_2/\gamma_1$; R and $\overline{R}$ are multiple power series having infinitely large radii of convergence and equal to one for $x' = x$ and $x' = y$. Moreover, if

$$R_{-\beta}(x'-x, y-x') = \sum_{\nu=0}^{\infty} \sum_{s=0}^{\infty} a_{\nu s} (x'-x)^\nu (y-x')^s \quad (a_{\nu s} = \text{const},\ a_{00} = 1), \tag{6}$$

then $R_{\beta-1}(x'-x, y-x')$ is obtained from series (6) by replacing β by $1-\beta$.

The functions $R_{-\beta}$ and $R_{\beta-1}$, which are strictly positive in the half-plane $y > x$, satisfy the inequalities

$$R_{-\beta}(P) \leq \overline{I}_{-\beta}(br), \qquad R_{\beta-1}(P) \leq \overline{I}_{\beta-1}(br), \tag{7}$$

$$r = \sqrt{(x'-x)(y-x')}, \qquad P = P(x'-x, y-x'), \qquad b = 2\sqrt{\sup_{y>x}(c/(y-x))}.$$

Analogously to the case $c = (\frac{1}{2}b)^2(y-x)$, where[2] $\overline{P} = P(x-x', y-x')$,

$$\overline{R}_{-\beta}(\overline{P}) = \overline{J}_{-\beta}(br_1), \qquad \overline{R}_{\beta-1}(\overline{P}) = \overline{J}_{\beta-1}(br_1), \qquad r_1 = \sqrt{(x-x')(y-x')}, \tag{8}$$

the series $\overline{R}_{-\beta}$ and $\overline{R}_{\beta-1}$ give oscillating functions possessing an infinite number of nodal lines for $y > x$. From Theorems 1 and 2 it follows:

1. Denote by $u_1(x, y, x_0, y_0)$ the Riemann function of equation (1), and by $H(x, y, x_0, y_0)$ and $\overline{H}(x, y, x_0, y_0)$ the Hadamard functions, respectively, of problems T_1 and T_2. Fix in the half-plane $y > x$ a point $C(x_0, y_0)$ (the observation point) and draw through it two characteristics CA and CB (the incident characteristics). At the points $A(x_0, x_0)$ and $B(y_0, y_0)$ (reflection points) construct the straight lines $y = x_0$ and $x = y_0$ (the reflected characteristics). Then we obtain six regions: $1(x_0 < y_0 < x < y)$, $2(x < x_0 < y_0 < y)$, $3(x < y < x_0 < y_0)$, $4(x_0 < x < y_0 < y)$, $5(x < x_0 < y < y_0)$, and $6(x_0 < x < y < y_0)$. By its initial values on the incident characteristics the function u_1 is defined in regions 2, 4, 5, and 6. On the other hand, the known initial data on the reflected characteristic and the line $y = x$, associated with H and $\overline{H}$ (4), determine these functions in region 1 (or 3).

As formulas (4) and (5) show, the initial values $u_1|_{y=x}$, $u_{1\xi}|_{y=x}$, $H|_{y=x}$, $\overline{H}_\xi|_{y=x}$, after removal of the multiplicative power singularities at the reflection points, become entire holomorphic functions. Solving with their help problem (1), (2), we obtain integral representations for the functions u_1, H, and $\overline{H}$ in regions 2, 6, 1 (or 3). Thus, for example, in region 6

$$u_1 = m(y-x)^{1-a}(y_0-x_0)^a \int_x^y r^{a-2} r_0^{-a} R_{-\beta}(P_0) R_{\beta-1}(P)\, dx' -$$

$$- m(y_0-x_0) \int_x^y r^{-a} r_0^{a-2} R_{-\beta}(P) R_{\beta-1}(P_0)\, dx' = u_{36} - u_{46}; \tag{9}$$

in region 3

$$H = n(y-x)^{1-a}(y_0-x_0)^a \int_x^y r^{a-2} r_{10}^{-a} R_{\beta-1}(P) \overline{R}_{-\beta}(\overline{P}_0)\, dx', \tag{10a}$$

$$\overline{H} = n(y_0-x_0) \int_x^y r^{-a} r_{10}^{a-2} R_{-\beta}(P) \overline{R}_{\beta-1}(\overline{P}_0)\, dx'. \tag{10b}$$

In this case

$$P_0 = P_0(x'-x_0,\ y_0-x'), \quad \overline{P}_0 = \overline{P}_0(x_0-x',\ y_0-x'),$$

$$r_0 = \sqrt{(x'-x_0)(y_0-x')}, \quad r_{10} = \sqrt{(x_0-x')(y_0-x')}, \quad m = \operatorname{tg}\beta\pi/2\pi.$$

The singular nature of the expressions (9) and (10), which are valid only in a neighborhood of the singular line $y = x$, necessarily leads to the problem of analytically continuing them into a neighborhood of the incident and reflected characteristics. For this purpose, along with the solutions $u_1, H, \overline{H}$, it is necessary to introduce new auxiliary principal solutions, whose number and the conditions serving for their unique construction are entirely determined by the number of singular lines and by the character of the singularity, on these lines, of the functions $u_1, u_3, u_4, H, \overline{H}$. Thus, for example, the integrals u_1, H, and $\overline{H}$ have logarithmic singularities on the reflected characteristics, near which these functions have the form

$$u = P \ln \Lambda + Q; \quad \Lambda = (y_0 - x)(y - x_0)/(x_0 - x)(y - y_0). \tag{11a}$$

Therefore the problem of analytic continuation of the Riemann and Hadamard functions into a neighborhood of the reflected characteristics requires the introduction of principal solutions u_5 and u_6, the first of which serves as the coefficient of $\ln \Lambda$ in formula (11a), while the second contains the indicated logarithmic singularity of the functions $u_1, H, \overline{H}$, but has a regular part $Q_6(x, y, x_0, y_0)$ vanishing on the reflected characteristics.

The functions u_{5k} $(k = 1, 3, 4, 5)$ are uniquely determined by their values on the reflected characteristics, analogous to the known initial data of the Riemann function on the incident characteristics. These functions, having logarithmic singularities on the incident characteristics, give in regions 4 and 5 fundamental solutions of equation (1) that are bounded on the reflected characteristics. Analogously, for the investigation of the fundamental solutions u_{3k} and u_{4k} $(k = 2, 6)$ in a neighborhood of the incident characteristics, where

$$u_{ik} = P_{ik}(x, y, x_0, y_0) \ln \Lambda^{-1} + Q_{ik}(x, y, x_0, y_0) \quad (i = 3, 4;\ k = 2, 6), \tag{11b}$$

it is necessary to construct branches u_{1k} $(k = 2, 4, 5, 6)$ of the Riemann function belonging to the incident characteristics in regions $2, 4, 5, 6$, and coinciding (up to a constant factor) with the coefficient of $\ln \Lambda^{-1}$ in formulas (11b), and also to introduce fundamental solutions u_{2k} $(k = 2, 4, 5, 6)$, carrying the logarithmic singularity of the functions u_{ik} $(i = 3, 4, 5, 6)$ on the incident characteristics, but having there zero regular parts Q_{2k}. The fundamental solutions u_{5k}, just like u_{3k}, u_{4k}, are constructed by combining singularities of the initial functions in the problems K, T_1, and T_2 with power singularities of the kernels in the solutions of these problems. As a result, for the functions u_{1k}, u_{5k} one obtains integral expressions analogous to those indicated earlier ([2]) for $c = (1/2b)^2(y - x)$. However, the products of Bessel functions under the integral signs of such expressions are replaced, in case (1), by products of power series of the form R_ν or $\overline{R}_\nu$.

Having obtained the branches u_{1k}, u_{5k}, bounded respectively on the incident and reflected characteristics, we can use them to construct the branches u_{2k}, u_{6k}, which have logarithmic singularities simultaneously on both the incident and the reflected characteristics. To this end, considering the inhomogeneous differential equations satisfied by the regular parts Q_{2k} and Q_{6k} of the integrals u_{2k} and u_{6k}, we solve for the functions Q_{2k} and Q_{6k} the Goursat problem with zero data respectively on the incident and reflected characteristics. The linear relations connecting each three of the four branches u_{ik}, belonging to regions 1 –6, analytically continue these branches from a neighborhood of one singular line into a neighborhood of the other singular line of the functions u_{ik}. The same

for the purpose of regular continuation from a neighborhood of one characteristic to a neighborhood of another there also serve two-term linear relations (symmetry relations) for the solutions u_{ik}. For example, the analytic continuation of the function u_5 from a neighborhood of the reflected characteristic $y_0 = x$ of domain 1 into a neighborhood of the transition line $y_0 = x_0$ gives the equality

$$\begin{aligned} n(y-x)^{1-a} \int_{y_0}^{x} r_{10}^{-a} r_1^{a-2}\, \overline{R}_{-\beta}(\overline{P}_0)\, \overline{R}_{\beta-1}(\overline{P})\, dx' \\ = m(y_0-x_0)^{1-a} \int_{x_0}^{y_0} r_1^{-a} r_0^{a-2}\, \overline{R}_{-\beta}(\overline{P})\, R_{\beta-1}(P_0)\, dx' \\ - m(y-x)^{1-a} \int_{x_0}^{y_0} r_1^{a-2} r_0^{-a}\, \overline{R}_{\beta-1}(\overline{P})\, R_{-\beta}(P_0)\, dx'. \end{aligned} \tag{12}$$

2. The functions R_ν and $\overline{R}_\nu$, equal to unity respectively on the incident and reflected characteristics, can be constructed from these initial data of Goursat by the usual methods of iterations or power series. Their effective computation makes it possible to obtain expansions of the principal solutions u_{ik} of equation (1) in uniformly and absolutely convergent series in Appell hypergeometric functions $F_1(\alpha, \beta, \beta', \gamma; X, Y)$ ([5]).

3. In the case under consideration $0 < a < 1$, the logarithmic singularities of the fundamental solutions (11b) disappear when the pole of these solutions falls on the transition line. This shows that for $0 < a < 1$ there do not exist solutions of equation (1) with logarithmic singularities at points of the transition line. Such solutions appear only for $a = \pm 1, \pm 2, \pm 3, ...$, when $y = x$ becomes a branching line of logarithmic character. Thus, for example, if $a = 1$, then in order to obtain the general integral of the form (4), and also in order to construct the principal solutions u_{ik} of equation (1), it is necessary to use the solutions of equation (1)

$$z_1(a,x,y) = \left[(x'-x)(y-x')\right]^{-a/2} R_{-a/2}(x'-x, y-x'),$$

$$z_2 = \lim_{a \to 1} \frac{z_1(a, x, y) - (y - x)^{1-a} z_1(2 - a, x, y)}{1 - a}.$$

4. The results obtained can be carried over to linear equations of higher orders and to systems of such equations with one or several regular singular lines. The equation

$$(\zeta - a) z_{\theta\theta} + (\zeta - b) z_{\zeta\zeta} = 0 \qquad (a < b),$$

deserves attention; it is a prototype of the well-known Chaplygin equations ([6,7]). The nature of the branching of the principal solutions of such an equation is complicated because of the multiple reflection of the incident characteristics from two parabolicity lines and the effect of interference of the singular lines.

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Note: Figure translations are in progress. See original paper for figures.