ON THE APPLICATION OF THE METHOD OF MOMENTS AND THE “MIXED” METHOD TO THE APPROXIMATE SOLUTION OF SINGULAR INTEGRAL EQUATIONS

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Abstract

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ON THE APPLICATION OF THE METHOD OF MOMENTS AND THE “MIXED” METHOD TO THE APPROXIMATE SOLUTION OF SINGULAR INTEGRAL EQUATIONS

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Below a justification is given of the two methods indicated in the title. Only the so-called “characteristic” equation is investigated in detail. The transition to the case of a general singular equation can be made without special difficulty according to the scheme developed by L. V. Kantorovich (^{1}).

Thus, let the equation be given

$$G\varphi \equiv A(t_0)\varphi(t_0) + \frac{B(t_0)}{\pi i}\int_\gamma \frac{\varphi(t)\,dt}{t-t_0} = f(t_0), \qquad t_0 \in \gamma, \tag{1}$$

where γ is the unit circle with center at the origin; $A, B, f \in H$, i.e. are Hölder-continuous on γ; $A^2 - B^2 \neq 0$ on γ; the index

$$\varkappa = \frac{1}{2\pi}\left[\arg\frac{A-B}{A+B}\right]_\gamma \geq 0.$$

Assume also that $A+B \equiv 1$; this does not restrict the generality of the reasoning.

1. Method of moments

We shall regard equation (1) as a linear equation in the Hilbert space L_2 (^{2}). For each function $\varphi \in L_2$, denote by φ^+ and φ^- the functions analytic respectively inside and outside γ and connected with φ by the Plemelj-Sokhotski formulas (^{2,3}). Under the assumptions made there exists (^{2,3}) a unique solution of equation (1), φ_*, for which φ_*^- has the highest order of zero at infinity in comparison with other solutions. We shall seek an approximate value φ_* in the form

$$\varphi_*^{(1)} = \sum_0^n \alpha_k t^k - \sum_{-n}^{-1} \alpha_k t^{k-\varkappa}.$$

Requiring that the scalar products $(G\varphi_*^{(1)}, t^k)$ and (f, t^k) coincide for all k, $-n \leq k \leq n$, we arrive at an algebraic system of equations for the unknowns $\{\alpha_k\}$:

$$\overline{G}\alpha = \int_\gamma \left\{ \sum_0^n \alpha_k t^k - [A(t) - B(t)]t^{-\varkappa} \sum_{-n}^{-1} \alpha_k t^k \right\} \bar{t}^{j+1}\, dt = \int_\gamma f(t)\bar{t}^{j+1}\, dt, \quad (2)$$

$$j = 0, \ \pm 1, \ldots, \pm n.$$

It is convenient to regard system (2) as a linear equation in the space $\overline{L}_2$, whose elements are collections of $2n+1$ numbers $\alpha\{\alpha_{-n}, \ldots, \alpha_n\}$, with norm

$$\|\alpha\|_{\overline{L}2} = \|\alpha_{-n}t^{-n} + \cdots + \alpha_n t^n\|_{L_2}.$$

Lemma 1. Let $Q_n = 1 + q_1 t^{-1} + \cdots + q_n t^{-n}$ have zeros inside γ, and let $P_n = p_0 + p_1 t + \cdots + p_n t^n$ have no zeros in common with Q_n. In that case the system

$$\overline{G}_1\bar{\alpha} = \int_\gamma \left(\sum_0^n \alpha_k t^k - \frac{P_n}{Q_n} \sum_{-n}^{-1} \alpha_k t^k \right) \bar{t}^{j+1}\, dt = \int_\gamma f(t)\bar{t}^{j+1}\, dt, \quad (3)$$

$$j = 0, \ \pm 1, \ldots, \pm n,$$

is uniquely solvable for any n.

Proof. Construct the polynomial $R_n = r_{-n}t^{-n} + \cdots + r_n t^n$ from the condition $(R_n/Q_n, t^k) = (f, t^k)$, $-n \leq k \leq n$. It is easy to see that such a construction is possible and is carried out in a unique way. Using the theorem on the expansion of a rational fraction into a sum of partial fractions, we find $\{\alpha_k\}$ such that the identity

$$Q_n(\alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n) - P_n(\alpha_{-1}t^{-1} + \cdots + \alpha_{-n}t^{-n}) \equiv R_n$$

is satisfied. The rest is obvious.

Lemma 2. Beginning with some $n \geq C_1$, system (2) has a unique solution, and

$$\left\|\varphi_* - \varphi_*^{(1)}\right\|_{L_2} \leq C_2 \left\|\varphi_* - T_n^{(1)}\right\|_{L_2}, \quad (4)$$

where $T_n^{(1)} = \tau_{-n}^{(1)}t^{-n} + \cdots + \tau_n^{(1)}t^n$ is the polynomial of best approximation to φ_* in the metric L_2, and the constants C_1 and C_2 (as also the following constants $C_3, C_4, \ldots$) do not depend on n and can be easily computed.

Proof. Let $\chi = \ln[(A-B)t^{-\varkappa}]$. Then

$$(A-B)t^{-\varkappa} = \frac{\exp(\chi^+)}{\exp(\chi^-)}.$$

Approximating $\exp(\chi^+)$ and $\exp(\chi^-)$ in the best way respectively by the polynomials P_n and Q_n, we find an n such that the zeros of P_n lie outside γ, and the zeros of Q_n inside γ. We apply the preceding lemma to P_n and Q_n. Then it will follow from Banach' s theorem (^{4}) that $\overline{G}_1$ has an inverse operator $\overline{G}_1^{-1}$, whose norm is bounded by a constant independent of n*.

Next, considering the difference of the operators $\overline{G} - \overline{G}_1$, it is easy to conclude that the operator $\overline{G}$ as well, beginning with some $n \geq C_1$, has an inverse operator $\overline{G}^{-1}$, whose norm remains bounded as n grows. After this the proof of the lemma is completed by a well-known device.

In the case of the general singular equation

$$K\varphi = G\varphi + \lambda \int_\gamma T(t, t_0)\varphi(t)\, dt = f(t_0), \qquad t_0 \in \gamma, \tag{5}$$

where λ is a complex parameter and T satisfies a Hölder condition in both variables, one can, relying on Lemma 2 and on a theorem of L. V. Kantorovich (^{1}) on the approximate solution of linear equations of the type $Gx + \lambda Tx = y$ (in which G has an inverse and T is a completely continuous operator), prove the following result.

Theorem 1. If, in the notation adopted earlier, equation (5) has a unique solution φ_*, then the system

$$(K\varphi_*^{(1)}, t^j) = (f, t^j), \qquad j = 0, \ \pm 1, \dots, \pm n, \tag{6}$$

has a unique solution for all $n \geq C_3$, and

$$\left\|\varphi_* - \varphi_*^{(1)}\right\|_{L_2} \leq C_4 \left\|\varphi_* - T_n^{(1)}\right\|_{L_2}. \tag{7}$$

In view of the dependence of the smoothness of the solution φ_* on the smoothness of the kernel and of the coefficients of equation (5), it is not difficult to derive from inequality (7) a number of consequences concerning the rate of convergence and to obtain effective error estimates (^{5}, ^{6}).

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* Practically, it is convenient to find an estimate for $\|\overline{G}_1^{-1}\|$ by using the solution of the Riemann problem Q_n

2. Mixed method

On the set of functions $\varphi \in H$ we introduce a norm by the rule $\|\varphi\| = \max_{t\in\gamma}|\varphi^+| + \max_{t\in\gamma}|\varphi^-|$. The closure of H in the sense of the introduced metric will be denoted by W^*. We shall regard equation (1) as a linear equation in the space W. We shall seek the approximate value $\varphi_*^{(2)}$ in the form

$$\varphi_*^{(2)} = \sum_0^n \alpha_k t^k - \sum_{-n}^{-1} \alpha_k t^{k-\chi}.$$

Requiring that $G\varphi_*^{(2)}$ be equal to f at $2n+1$ points $\{t_j\}$, which divide γ into $2n+1$ equal parts, we arrive at the algebraic system:

$$\overline{G}\,\overline{\alpha} \equiv \sum_0^n \alpha_k t_j^k - [A(t_j) - B(t_j)]t_j^{-\chi-1} \sum_{-n}^{-1} \alpha_k t_j^k = f(t_j) \equiv f_j, \tag{8}$$

$$j = 1, 2, \dots, 2n+1.$$

We further introduce the operators ψ, ψ^{-1}, which establish a one-to-one correspondence between the elements $\alpha\{\alpha_{-n}, \dots, \alpha_n\}$ and $\overline{a}\{a_1, \dots, a_{2n+1}\}$ according to the law

$$a_j = \alpha_{-n}t_j^{-n} + \dots + \alpha_n t_j^n, \quad j = 1, 2, \dots, 2n+1.$$

The equation $N\overline{a} \equiv \overline{G}\psi^{-1}\overline{a} = \overline{f}$ will be regarded as a linear equation in the space $\overline{W}$, whose elements are sets of $2n+1$ numbers with norm

$$\|\overline{a}\|_{\overline{W}} = \|\alpha_{-n}t^{-n} + \dots + \alpha_n t^n\|_W, \quad \text{where } \overline{\alpha} = \psi^{-1}\overline{a}.$$

Lemma 3. The inequality

$$\|\alpha_{-n}t^{-n} + \dots + \alpha_n t^n\|_W \le C_5 \ln n \max_{1\le j\le 2n+1} \left|\alpha_{-n}t_j^{-n} + \dots + \alpha_n t_j^n\right|$$

is valid.

The **proof** almost completely coincides with the proof of the known fact (^{7}) that

$$|\alpha_{-n}t^{-n} + \dots + \alpha_n t^n| \le C_6 \ln n \max_{1\le j\le 2n+1} |\alpha_{-n}t_i^{-n} + \dots + \alpha_n t_i^n|\,.$$

Lemma 4 (^{8}). If $\varphi(t)$ satisfies on γ the Hölder condition with exponent μ, and T_n is a polynomial of degree n (in positive and negative powers of t) for which

$$|\varphi - T_n| < C_7/n^\mu, \quad t \in \gamma,$$

then T_n satisfies the Hölder condition with the same exponent and with a constant bounded as n grows.

Lemma 5. Suppose that the polynomials P_n and Q_n (see Lemma 1) have no common zeros and $Q_n \neq 0$ on γ. Then the system

$$\overline{G}_1\alpha \equiv \alpha_0 + \alpha_1 t_j + \dots + \alpha_n t_j^n - (P_n/Q_n)_j(\alpha_{-1}t_j^{-1} + \dots + \alpha_{-n}t_j^n) = f_j, \qquad (9)$$

$$j = 1, 2, \dots, 2n+1,$$

has a unique solution.

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* It is easy to show that W consists of all such functions φ for which φ^+ and φ^- are continuous.

Indeed, the solution of this system will be the $\{\alpha_k\}$ that satisfy the relation $Q_n(\alpha_0+\alpha_1 t+\dots+\alpha_n t^n)-P_n(\alpha_{-1}t^{-1}+\dots+\alpha_{-n}t^{-n}) = R_n$, where the polynomial R_n (see Lemma 1) is constructed according to the condition $R_n(t_j) = f_j Q_n(t_j)$, $j = 1, 2, \dots, 2n+1$. As a consequence of this lemma there follows the unique solvability of the equation $N_1 a \equiv G_1\psi^{-1}a = \bar{f}$.

Lemma 6. The system (8), beginning with some $n \geq C_8$, has a unique solution and, moreover,

$$\|\varphi_*^{(2)}\|_W < C_9 \ln n \max_{1\leq j\leq 2n+1} |f(t_j)|.$$

Proof. We take for P_n and Q_n the polynomials approximating $\exp(\chi^+)$ and $\exp(\chi^-)$ (see p. 1). We easily obtain that $\|N_1\|_W < C_{10}\ln n$. Using Lemmas 3, 4 and the exact formulas (3) for the solution of the Riemann boundary-value problem $Q_n\psi^+ - P_n\psi^- = R_n$, we find that $\|N_1^{-1}\|_W < C_{11}\ln n$. Comparing the operator N with the operator N_1 in the metric W, by elementary computations we arrive at the fact that, beginning with some n, the operator N has an inverse and that $\|N^{-1}\|_W < C_{12}\ln n$. From this fact there follows immediately what was required to be proved.

Relying on this result, one can prove the following theorem.

Theorem 2. If, in the previous notation, equation (5) has a unique solution φ_*, then the system

$$[K\varphi_*^{(2)}](t_j) = f(t_j), \qquad j = 1, 2, \dots, 2n+1, \qquad (10)$$

beginning with some $n \geq C_{13}$, also has a unique solution, and

$$\|\varphi_* - \varphi_*^{(2)}\|_W < C_{14}\ln n \max_{t\in\gamma} |\varphi_* - T_n^{(2)}|, \qquad (11)$$

where $T_n^{(2)} = \tau_{-n}^{(2)}t^{-n} + \dots + \tau_n^{(2)}t^n$ is the polynomial of best approximation to φ_* in C.

The investigation of equation (5) for stability in the space W shows: if the new elements of equation (5), A, B, f, and T, while remaining in H, differ from the original ones in the metric C by ε and have Hölder constants bounded as ε varies, and the original equation has a unique solution, then for sufficiently small ε the new equation will also have a unique solution, and the error of the solution in the metric W will be of order $\varepsilon \ln(1/\varepsilon)$.

Let us note that if the original singular integral equation is given on a finite collection of simple smooth arcs with continuous curvature, then by a change of variables one can reduce the matter to the case where the equation is given on a collection of arcs lying on γ. Applying the considerations set forth in (9), the method of moments and the "mixed" method, with certain modifications, can be adapted to this more general equation as well.

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Note: Figure translations are in progress. See original paper for figures.