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Abstract

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THE FIRST BOUNDARY-VALUE PROBLEM FOR A NONLINEAR ELLIPTIC EQUATION OF SECOND ORDER IN A BOUNDED DOMAIN WITH A DEGENERATE BOUNDARY

(Presented by Academician S. L. Sobolev, 28 I 1957)

1. In the present note we consider the formulation of the first boundary-value problem—the finding of a solution from its values on the boundary of a bounded domain—for a quasilinear elliptic equation that is the Lagrange equation for a certain variational problem. It turns out that, in contrast to the linear case, for a very broad class of nonlinear problems the boundary values may be prescribed on manifolds of any number of dimensions smaller than the dimension of the original space. For example, in the two-dimensional case the boundary may consist of a closed curve bounding the domain and a finite number of smooth curves and points inside it. The solutions obtained are continuous in such a domain up to the boundary and, for a smooth integrand, are solutions in the classical sense.

Let us note that the problem in a domain with a degenerate boundary was first considered by S. L. Sobolev (ˆ2)—for the polyharmonic equation—and then by V. I. Kondrashev (ˆ3) for variational problems with an integrand consisting of the sum of the p-th powers of the highest derivatives and a polynomial of degree not exceeding p in the lower derivatives. The differential properties of the solution in these works are investigated only when there is a fundamental solution, i.e. when the functional is quadratic with respect to the highest derivatives—but this is precisely the case in which the formulations of the boundary-value problem for the nonlinear and linear equations coincide. In the present work a problem is investigated for functionals of a more general form.

2. Consider the following example: let K be the unit disk in the plane with the center at the origin removed, and

$$J(u) = \iint\limits_K \{\alpha(u_x^2 + u_y^2)^2 + \beta(u_x^2 + u_y^2)\}\, dx\, dy. \tag{1}$$

We seek a function $u_0 \in W_4^{(1)}$ that assumes on the boundary of K the values

$$\begin{array}{ll} \varphi(\gamma) & \text{for } x^2+y^2=1, \\ \varphi_0 & \text{for } x^2+y^2=0, \end{array} \tag{2}$$

where $\varphi(\gamma)$ is admissible for $W_4^{(1)}$.

Before solving this problem, let us note that:

1) for $\alpha = 0$ the problem has no solution;
2) for $\beta > 0$, $\alpha > 0$ the problem is regular, i.e. for a smooth solution it reduces to a nondegenerate elliptic equation;
3) for $\varphi(\gamma) \equiv 1$ and $\varphi_0 = 0$ the solution $u(x, y) = u(\sqrt{x^2+y^2}) = u(\Gamma)$ satisfies the equation

$$u'(\Gamma)^3 + \gamma u'(\Gamma) = \frac{C}{\Gamma}, \qquad C > 0, \quad \gamma > 0, \tag{3}$$

and the conditions $u(0) = 0$, $u(1) = 1$ (it is easy to verify that such a solution exists, is unique, and is a function smooth inside the domain).

This consideration makes it possible, to a considerable extent, to anticipate the subsequent results.

From the embedding theorems $^{(1)}$ it follows that the class of solutions of problem (1), (2) consists of continuous functions and that convergence in it guarantees convergence in C. By $W_4^{(1)}(\varphi)$ we shall denote the set of functions from $W_4^{(1)}$ taking the boundary values (2). This class is nonempty, since $\varphi(\gamma)$ is admissible. Since $J(u) \geq 0$, there exists $d = \inf J(u)$ over $u \in W_4^{(1)}(\varphi)$ and, consequently, there exists a sequence $u_n \in W_4^{(1)}(\varphi)$ such that $J(u_n) \to \alpha$ as $n \to \infty$.

It is easy to show that $J(u)$ satisfies the requirement of uniform convexity, i.e.

$$J\left(\frac{u+v}{2}\right) + J\left(\frac{u-v}{2}\right) \leq \frac{1}{2}(J(u) + J(v)), \tag{4}$$

whence it follows immediately that

$$J\left(\frac{u_n - u_m}{2}\right) \to 0 \quad \text{as } m, n \to \infty. \tag{5}$$

From (2) it is immediately clear that then $\|u_n - u_m\|_{W^{(1)}4} \to 0$, i.e. $u_n \to u_0 \in W_4^{(1)}(\varphi)$ $^{(1)}$. Since $J(u)$ is continuous on $W_4^{(1)}$, $J(u_0) = d$.

Thus we have obtained a solution $u_0(x, y)$ of the variational problem, taking on the boundary K the prescribed values (2) and continuous in $\overline{K}$.

By the Shiffman-Sigalov method $^{(4)}$ one proves the existence, for $u_0(x, y)$, of generalized second derivatives summable to the fourth power (i.e. $u_0 \in W_4^{(2)}$) in any strictly interior domain, whence it follows that $u_0 \in C^{(1)}(\overline{L})$, $\overline{L} \subset K$. Let us note at once that the part of Sigalov' s work used by us does not have (unlike other conclusions of his work) a specifically two-dimensional character. From Morrey' s work (5) it follows that $u \in C^2(\overline{L})$.

Thus $u_0(x, y)$ is a smooth solution in K of the variational problem and, consequently, satisfies the Euler-Lagrange equation for it. From (4) the uniqueness of the solution of the variational problem is immediately clear.

Let us note that, for the two-dimensional case, the analyticity of the solution inside the domain follows from the results of S. N. Bernstein (6).

3. Theorem. Let Ω be a bounded domain in n-dimensional space with boundary S, consisting of manifolds of dimensions $n-1, \dots, 1, 0$, such that the embedding theorems are valid for them, and let the functional

$$J(u) = \int_\Omega F\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) d\Omega \geq 0$$

satisfy the following requirements:

A. There exists a number $p > n$ such that $J(u)$ is lower semicontinuous on $W_p^{(1)}$ in Ω and from $J(u_n) \to 0$ it follows that $\|u_n\|_{L^{(1)}p} \to 0$, where

$$\|u\|^p_{L^{(1)}p} = \int_\Omega \left[\sum_{k=1}^n \left(\frac{\partial u}{\partial x_k}\right)^2\right]^{p/2} d\Omega.$$

B. $J(u)$ is uniformly convex, i.e. satisfies inequality (4).

Then for any function $\varphi(s)$, prescribed on the boundary S and admissible for $W_p^{(1)}$, there exists a function $u_0 \in W_p^{(1)}(\varphi)$ (since $p > n$, $u_0 \in C(\overline{\Omega})$) giving $J(u)$ a minimum on the class $W_p^{(1)}(\varphi)$.

If, moreover, $F\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$ satisfies certain requirements sufficient for the applicability of the Shiffman-Sigalov method (4),

for example, one has

$$\left.\begin{aligned} F(p_1 \dots p_n) &\leq m\left[\sum_{i=1}^n (p_i)^2\right]^{p/2} \\ \sum_{i,k=1}^n \left|\frac{\partial^2 F}{\partial p_i \partial p_k}\right|^2 &\leq m\left[\sum_{i=1}^n p_i^2\right]^{p/2-1} \end{aligned}\right\} \quad \text{for } \sum_{i=1}^n p_i^2 > L; \tag{6}$$

$$\sum \frac{\partial^2 F}{\partial p_i \partial p_k} \xi_i \xi_k \geq m_0 \sum_{i=1}^{n} \xi_i^2 \quad \text{for any } \xi = (\xi_1, \dots, \xi_n), \tag{7}$$

then $u_0 \in W_p^{(2)}$ in any Ω_1, where $\overline{\Omega}_1$ lies strictly inside Ω; in particular, $u_0 \in C^{(1)}(\Omega_1)$. Hence it follows at once that u_0 satisfies the Lagrange equation in generalized derivatives. From Koshelev' s results[7] it follows that, for arbitrarily differentiable $F(p_1, \dots, p_n)$, $u_0 \in W_\infty^{(2)}$ in Ω_1.

For $n = 2$, as was said above, the results of Morrey and Bernstein are applicable.

4. **Remarks.**

1) For $p < n$ the theorem does not hold, as is shown by the example

$$J(u) = \iint_K \left[1 + \left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2\right]^{p/2} dx\, dy, \qquad 1 < p \leq 2,$$

where K is the unit disk, and the boundary values are

$$\varphi_1 \equiv 1 \quad \text{for } x^2 + y^2 = 1,$$

$$\varphi_0 = 0 \quad \text{for } x^2 + y^2 = 0.$$

2) Our theorem is based on the following elementary geometric theorem of functional analysis.

Suppose that in a complete metric linear space there is a closed convex set not containing zero. Then the distance between zero and this set is attained at a point of the set, and this point is unique (the metric is assumed to be uniformly convex).

3) The theorem admits obvious generalizations to the case of F depending on the independent variables and the function, and admits a considerable weakening of other requirements.

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CITED LITERATURE

1. S. L. Sobolev, *Some applications of functional analysis in mathematical physics*, L., 1950.

2. S. L. Sobolev, *Matem. sborn.*, **2** (44), issue 3 (1937).

3. V. I. Kondrashov, *Theory of boundary value problems*, Doctoral dissertation, Steklov Mathematical Institute, Academy of Sciences of the USSR, 1948; *DAN*, **90**, No. 2 (1953).

4. A. G. Sigalov, *Tr. Moskov. matem. obshch.*, **2** (1953).

5. C. B. Morrey, *Trans. Am. Math. Soc.*, **43** (1) (1938).

6. S. N. Bernstein, *Uspekhi matem. nauk*, **8**, 32 (1940).

7. A. I. Koshelev, *DAN*, **105**, No. 1 (1955).

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