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Abstract

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Yu. G. LUMISTE

ON THE GEOMETRIC STRUCTURE OF A COMPLEX-ANALYTIC SURFACE V_{2n} IN THE SPACE R_{2N}

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1. A **complex-analytic surface** V_{2n} in Euclidean space R_{2N} is a surface which, in orthogonal coordinates $x^J, x^{\bar J}$ $(J = 1, \dots, N;\ \bar J = N + J)$, can be represented by the equations

$$x^J + ix^{\bar J} = f^J(u^k + iu^{\bar k}) \qquad (k = 1, \dots, n),$$

where $f(w^k)$ are analytic functions of n complex variables $w^1, \dots, w^n$.

In the case $n = 1$, such surfaces were considered by Kommerell and Eisenhart (1) for $N = 2$, and by Borůvka (2) for arbitrary N. They proved that V_2 in R_{2n} is complex-analytic if and only if all its indicatrices of normal curvature are circles. In this case the surface is a surface of translation of its isotropic curves.

In the present article a geometric characterization is given of a complex-analytic V_{2n} in R_{2N}.

2. The complex-analytic V_{2n} in R_{2N} is considered as the image of an analytic surface W_n of the unitary space $U_N(i)$ under an isometric mapping $U_N(i) \to R_{2N}$, under which the vector $\xi^J e_J \in U_N(i)$ is mapped into the real vector $x^J e_J + x^{\bar J} e_{\bar J} \in R_{2N}$ with $x^J = x^{\bar J} = \xi^J$, where $e_J, e_{\bar J} = e_J$ are vectors in R_{2N} lying, respectively, in two imaginary complex-conjugate completely isotropic directions $I_N, \overline{I}_N$, so that $(e_J, e_{\bar K}) = (\varepsilon_J \varepsilon_K)$. The image of the line $\theta(\xi^J e_J) \in U_N(i)$, passing through the initial point of the frame, is the two-dimensional complex-analytic plane $(\theta\xi^J)e_J + (\overline{\theta\xi^J})e_{\bar J} \in R_{2N}$, determined by two complex-conjugate vectors in the directions $I_N, \overline{I}_N$.

An analytic moving frame in $U_N(i)$, whose infinitesimal displacement formulas are

$$dM = \pi^J e_J, \qquad de_J = \pi_J^K e_K,$$

is mapped into a frame whose displacement is determined by the formulas

$$dM = \omega^J e_J + \omega^{\bar J} e_{\bar J}, \qquad de_J = \omega_J^K e_K, \qquad de_{\bar J} = \omega_{\bar J}^{\bar K} e_{\bar K},$$

where

$$\omega^{\bar J} = \overline{\omega^J} = \overline{\pi^J}, \qquad \omega_{\bar J}^{\bar K} = \overline{\omega_J^K} = \overline{\pi_J^K}, \tag{1}$$

i.e., the image of the frame moves in such a way that

$$\omega_J^{\bar K} = \omega_{\bar J}^K = 0. \tag{2}$$

3. **Theorem.** *A real non-isotropic surface V_{2n} in the Euclidean space R_{2N} is complex-analytic if and only if:*

1°. It is the transfer surface of two imaginary complex-conjugate completely isotropic analytic surfaces $X_n, \overline{X}_n$.

2°. The surfaces $X_n, \overline{X}_n$ lie respectively in two flat generators $I_N, \overline{I}_N$ of the isotropic cone, intersecting only at a point of the surface.

A complex-analytic surface V_{2n} belongs, therefore, to the class of minimal surfaces with two isotropic conjugate directions $I_n, \overline{I}_n$.

Necessity. Let the complex-analytic $V_{2n} \subset R_{2N}$ be the image of the analytic surface $W_n \subset U_N(i)$ under the mapping $U_N(i) \to R_{2N}$. An analytic moving frame is attached to the point M of the surface W_n so that the vectors e_i $(i, j, \ldots = 1, \ldots, n)$ lie in the tangent plane to W_n at the point M. Then $\pi^\alpha = 0$ $(\alpha, \beta, \ldots = n+1, \ldots, N)$. Prolongation of these equations leads to the equations $\pi_i^\alpha = \Lambda_{ij}^\alpha \pi^j$.

The moving frame attached to V_{2n} under the mapping $U_N(i) \to R_{2N}$ moves, by virtue of (1) and (2), in such a way that

$$\omega_i^{\bar j} = \omega_{\bar i}^j = \omega_i^{\bar\alpha} = \omega_{\bar i}^\alpha = \omega_\alpha^{\bar\beta} = \omega_{\bar\alpha}^\beta = 0,$$

$$\omega_i^\alpha = \Lambda_{ij}^\alpha \omega^j, \qquad \omega_{\bar i}^{\bar\alpha} = \Lambda_{\overline{ij}}^{\bar\alpha} \omega^{\bar j}.$$

The systems $\omega^i = 0$, $\omega^{\bar i} = 0$ will both be completely integrable, and the directions $I_n, \overline{I}_n$, constructed on the vectors $e_i, e_{\bar i}$, envelop families of completely isotropic sub-surfaces X_n and $\overline{X}_n$.

The necessity of conditions 1° and 2° now follows from the formulas

$$de_i = \omega_i^j e_j + \Lambda_{ij}^\alpha \omega^j e_\alpha, \qquad de_\alpha = \omega_\alpha^i e_i + \omega_\alpha^\beta e_\beta,$$

$$de_{\bar i} = \omega_{\bar i}^{\bar j} e_{\bar j} + \Lambda^{\bar\alpha}_{\bar i\bar j}\omega^{\bar j} e_{\bar\alpha}, \qquad de_{\bar\alpha} = \omega_{\bar\alpha}^{\bar i} e_{\bar i} + \omega_{\bar\alpha}^{\bar\beta} e_{\bar\beta}.$$

Sufficiency. The moving frame at the point M of the surface $V_{2n} \subset R_{2N}$ is attached so that the vectors e_i, $e_{\bar j} = \bar e_j$ are tangent respectively to the sub-surfaces $X_n, \overline{X}_n$, while e_α, $e_{\bar\alpha}$ lie in the planes $I_N, \overline{I}_N$ and are orthogonal respectively to $e_{\bar i}, e_i$. Then $g_{ij} = g_{\bar i\bar j} = g_{\alpha\beta} = g_{\bar\alpha\bar\beta} = g_{i\alpha} = g_{\bar i\bar\alpha} = g_{i\bar\alpha} = g_{\bar i\alpha} = 0$, and

$$\omega^\alpha = \omega^{\bar\alpha} = 0. \tag{3}$$

From condition 1° it follows that, in the formulas of the infinitesimal displacement of the frame, the forms $\omega_i^{\bar j}, \omega_i^{\bar\alpha}, \omega_\alpha^{\bar i}, \omega_\alpha^{\bar\beta}$ and $\omega_{\bar i}^{j}, \omega_{\bar i}^{\alpha}, \omega_{\bar\alpha}^{i}, \omega_{\bar\alpha}^{\beta}$ are expressed only, respectively, in terms of $\omega^{\bar k}$ and ω^k. From condition 2° it follows that the same forms are expressed only, respectively, in terms of ω^k and $\omega^{\bar k}$. Consequently,

$$\omega_i^{\bar j} = \omega_i^{\bar\alpha} = \omega_\alpha^{\bar i} = \omega_\alpha^{\bar\beta} = \omega_{\bar i}^{j} = \omega_{\bar i}^{\alpha} = \omega_{\bar\alpha}^{i} = \omega_{\bar\alpha}^{\beta} = 0. \tag{4}$$

Exterior differentiation of equations (3) leads to the equations

$$[\omega^i \omega_i^\alpha] = 0, \qquad [\omega^{\bar i}\omega_{\bar i}^{\bar\alpha}] = 0. \tag{5}$$

The system of Pfaffian equations

$$dM = \omega^i e_i, \qquad de_i = \omega_i^j e_j + \omega_i^\alpha e_\alpha, \qquad de_\alpha = \omega_\alpha^i e_i + \omega_\alpha^\beta e_\beta,$$

where $\omega^i, \omega_i^j, \omega_i^\alpha, \omega^\alpha, \omega_\alpha^\beta$ are prescribed forms of an infinitesimal displacement of the frame attached to V_{2n}, now turns out, by virtue of (4), (5) and the structural equations of the space R_{2N}, to be completely integrable and defines a certain n-dimensional surface W_n in the space $U_N(i)$ with metric tensor

$$\gamma_{ij} = \bar\gamma_{ji} = g_{ij}, \qquad \gamma_{i\alpha} = 0, \qquad \gamma_{\alpha\beta} = \bar\gamma_{\beta\alpha} = g_{\alpha\bar\beta}.$$

Since $D\omega^i = [\omega^j\omega_j^i] \equiv 0 \pmod{\omega^k}$, we have $\omega^i = a_k^i\, dw^k$, and $dM = M_k\, dw^k$, where $M_k = a_k^i e_i$. Consequently, $\partial M/\partial \overline{w}^k = 0$, and the radius vector of a point of the surface W_n is an analytic function of n complex parameters w^k. The image of the surface W_n under the mapping $U_N(i) \to R_{2N}$ is the surface V_{2n} under consideration.

4. A question arises as to the relation between the theorem and the result of Kommerell and Eisenhart.

Let a one-dimensional direction tangent to the complex-analytic V_{2n} in R_{2N} rotate in a complex-analytic two-dimensional direction. It follows from (3) that the endpoint of the corresponding vector of normal curvature describes a circle lying in a complex-analytic two-dimensional direction of the normal plane.

Thus the complex-analytic surface V_{2n} satisfies a certain condition which is a generalization of the Kommerell–Eisenhart condition. However, this condition is not sufficient in the general case.

In the particular case when $N = n + 1$ and the normal plane of the surface V_{2n} has a positive-definite metric, condition 2° of the theorem can nevertheless be replaced by the following condition:

$2'$. *When the tangent direction is rotated in the two-dimensional direction determined by two complex-conjugate directions tangent, respectively, to* $X_n, \overline{X}_n$, *the endpoint of the corresponding vector of normal curvature describes a circle in the two-dimensional normal plane.*

Let the vectors $e_i, e_{\bar{i}} = \overline{e}_i, e_N, e_{\bar{N}}$ of the moving frame attached to a point of the surface V_{2n} be directed so that

$$g_{ij} = g_{\bar{i}\bar{j}} = g_{NN} = g_{\bar{N}\bar{N}} = g_{iN} = g_{i\bar{N}} = g_{\bar{i}N} = g_{\bar{i}\bar{N}} = 0, \tag{6}$$

$$\Lambda^{N}_{\bar{i}j} = \Lambda^{\bar{N}}_{i\bar{j}} = 0.$$

From condition 1° it follows that $\omega_i^{\bar{j}} = \Gamma^{\bar{j}}_{ik}\omega^k$, $\Gamma^{\bar{j}}_{ik} = \Gamma^{\bar{j}}_{ki}$. If this expression is substituted in the differential consequence of (6_1), one obtains the equation

$$g_{i\bar{l}}\Gamma^{\bar{l}}_{jk} + g_{l\bar{j}}\Gamma^{\bar{l}}_{ik} = 0.$$

Cyclic permutation of this equation in the indices i, j, k gives three equations, from which it follows that $\Gamma^{\bar{j}}_{ik} = 0$, i.e. $\omega_i^{\bar{j}} = 0$. Analogously it is proved that $\omega_{\bar{i}}^{j} = 0$. Differentiation of $(6_{3,4})$ leads to the results

$$\omega^{N}_{\bar{N}} = \omega^{\bar{N}}_{N} = 0.$$

Since the vector of normal curvature corresponding to the direction $\omega^i = \theta\xi^i$, $\omega^{\bar{i}} = \theta\bar{\xi}^i$ is equal to Φ/ds^2, where

$$\Phi = (\varphi^N\theta^2 + \psi^N\bar{\theta}^2)e_{\bar{N}} + (\varphi^{\bar{N}}\bar{\theta}^2 + \psi^{\bar{N}}\theta^2)e_N,$$

$$\varphi^N = \Lambda^N_{ij}\xi^i\xi^j, \qquad \psi^N = \Lambda^N_{\bar{i}\bar{j}}\bar{\xi}^i\bar{\xi}^j, \qquad \varphi^{\bar{N}} = \Lambda^{\bar{N}}_{\bar{i}\bar{j}}\bar{\xi}^i\bar{\xi}^j, \qquad \psi^{\bar{N}} = \Lambda^{\bar{N}}_{ij}\xi^i\xi^j,$$

then condition $2'$ is written in the form of the identity

$$(\Phi, \Phi) \equiv C\, ds^4 \tag{7}$$

with respect to $\theta, \bar{\theta}$, where C does not depend on $\theta, \bar{\theta}$.

From the reality of the vector Φ and the identity (7) it follows that $\psi^N \equiv \bar{\varphi}^N \equiv 0$ identically with respect to ξ^i, i.e. $\Lambda_{ij}^N = \Lambda_{ij}^{\bar{N}} = 0$, and, consequently, $\omega_N^i = \omega_{\bar{N}}^i = \omega_i^{\bar{N}} = \omega_i^N = 0$.

It is now not difficult to verify that condition 2° is satisfied.

Moscow State University
named after M. V. Lomonosov and
Tartu State University

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