ASYMPTOTIC BEHAVIOR OF THE SPECTRAL MATRIX OF THE OPERATOR OF THE THEORY OF ELASTICITY

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

M. Sh. FLEKSER

ASYMPTOTIC BEHAVIOR OF THE SPECTRAL MATRIX OF THE OPERATOR OF THE THEORY OF ELASTICITY

(Presented by Academician M. V. Keldysh on 28 II 1957)

In the works of B. M. Levitan ([1]) and V. A. Marchenko ([2]) new methods are indicated for the investigation of the spectral properties of differential operators. In the present note these methods are applied to the consideration of the question of the asymptotic behavior of the elements of the spectral matrix of the operator of the theory of elasticity, given in a finite domain.

The formulas obtained refine one result of A. Pleijel ([3]), which was obtained anew by another method by T. V. Burchuladze ([4]).

1. Denote by Ω a finite domain of three-dimensional Euclidean space E_3. Let $f(p) = \{f^1(p), f^2(p), f^3(p)\}$ (p is a point of Ω) denote a vector whose components have continuous derivatives up to the second order inclusive.

Consider the Cauchy problem:

Find a solution of the equation

$$-\Delta^* v + \frac{\partial^2 v}{\partial t^2} = 0, \qquad v = \{v^1, v^2, v^3\} \tag{1}$$

under the initial conditions

$$v(p, 0) = 0, \qquad \frac{\partial v(p, 0)}{\partial t} = f(p), \tag{2}$$

where $\Delta^* = b^2\Delta + (a^2 - b^2)\,\mathrm{grad}\,\mathrm{div}$ ($a > b > 0$ are constants, Δ is the Laplace operator).

If the point p lies inside Ω and t is so small that the ball with center at the point p and radius at lies entirely inside Ω, then the solution of the Cauchy problem can be represented in the form ([5])

$$4\pi v(p, t) = \frac{1}{b^2}\frac{\partial}{\partial t}\int_{r<bt} \frac{f(q)}{r}\, dq + \mathrm{grad}\,\mathrm{div} \int_0^t d\nu \int_{b\nu<r<a\nu} \frac{f(q)}{r}\, dq, \tag{3}$$

where $r = |p - q|$ is the distance between the points p, q; dq is the element of volume.

2. Consider some self-adjoint operator $\mathcal{L}$, generated by the differential operation $-\Delta^*$ in the domain Ω, and the eigenvalue problem

$$\mathcal{L}u - \mu^2 u = 0, \qquad u = \{u^1, u^2, u^3\} \tag{4}$$

for this operator.

Let μ_n^2 $(n = 1, 2, ...)$ be the eigenvalues of problem (4), and $u_n(p) = \{u_n^1(p), u_n^2(p), u_n^3(p)\}$ the corresponding eigenvectors.

We shall call the matrix the **spectral matrix** of the operator $\mathcal{L}$ matrix

$\Theta(p, q; \mu)$, whose elements $\vartheta_{ik}(p, q; \mu)$ $(i, k = 1, 2, 3)$ are determined by the formulas

$$\vartheta_{ik}(p, q; \mu) = \sum_{\mu_n < \mu} u_n^i(p) u_n^k(q), \qquad p, q \in \Omega.$$

Theorem. Let h be an arbitrary positive number and let Ω_h be the set of points of the domain Ω whose distance from its boundary is not less than h. Let the points $p = (x_1, x_2, x_3)$ and q belong to Ω_h. Then for the elements of the spectral matrix $\Theta(p, q; \mu)$ the following asymptotic formulas hold:

$$\overline{\lim_{\mu\to\infty}} \frac{1}{\mu^2} \sup_{p,q\in\Omega_h} \left| \vartheta_{ik}(p, q; \mu) - \frac{1}{b^3}\vartheta\left(\frac{r}{b}; \mu\right)\delta_{ik} - \frac{1}{2\pi^2} \frac{\partial^2}{\partial x_i \partial x_k} \frac{\operatorname{si}\left(\frac{\mu r}{b}\right) - \operatorname{si}\left(\frac{\mu r}{a}\right)}{r} \right| < \frac{10^8(a^{-3} + 2b^{-3})}{3\pi^2 h},$$

where $\vartheta(r, \mu)$ is the spectral function of the Laplace operator in the whole space E_3 (1); δ_{ik} is the Kronecker symbol; $\operatorname{si}(x)$ is the integral sine; $r = |p - q|$.

Proof. Let $f(q)$ be a vector vanishing outside Ω_h and sufficiently smooth that the Cauchy problem (1), (2) can be solved by the Fourier method. Then, for sufficiently small values of t, the equality holds

$$4\pi \int_0^\infty \frac{\sin \mu t}{\mu} d_\mu \int_{E_3} \Theta(p, q; \mu) f(q)\, dq = \frac{1}{b^2} \frac{\partial}{\partial t} \int_{r<bt} \frac{f(q)}{r}\, dq + \operatorname{grad} \operatorname{div} \int_0^t dy \int_{bv<r<av} \frac{f(q)}{r}\, dq. \tag{5}$$

Continue the right-hand side of (5) oddly onto the negative half-axis $-\infty < t < 0$, and then multiply both sides of equality (5) by the derivative $g'(t)$ of an arbitrary infinitely differentiable even function $g(t)$ vanishing outside the interval $(-h, h)$.

Integrating by parts with respect to the variable t and representing the vector $f(q)$ on the right-hand side of (5) by a Fourier integral, we obtain:

$$\int_{-\infty}^{\infty} E_g(\mu)\, d_\mu \int_{E_3} \Theta(p,q;\mu) f(q)\, dq =$$

$$= \int_{-\infty}^{\infty} E_g(\mu)\, d\mu \int_{E_3} \left\{ \frac{1}{b^3} \vartheta\left(\frac{r}{b};\mu\right) f(q) + \frac{1}{2\pi^2} \operatorname{grad}_p \operatorname{div}_p \left[\frac{\operatorname{si}\left(\frac{\mu r}{b}\right) - \operatorname{si}\left(\frac{\mu r}{a}\right)}{r} f(q) \right] \right\} dq, \tag{6}$$

where $E_g(\mu)$ is the Fourier transform of the function $g(t)$; the matrix $\Theta(p,q;\mu)$ is assumed to have been extended oddly onto the negative half-axis $-\infty < \mu < 0$.

By virtue of the arbitrariness of the vector $f(q)$, equality (6) is equivalent to the nine basic relations:

$$\int_{-\infty}^{\infty} E_g(\mu)\, d_\mu [\vartheta_{ik}(p,q;\mu) - \gamma_{ik}(p,q;\mu)] = 0, \qquad i,k = 1,2,3, \tag{7}$$

where it is set that

$$\gamma_{ik}(p,q;\mu) = \frac{1}{b^3}\, \vartheta\left(\frac{r}{b};\mu\right) \delta_{ik} + \frac{1}{2\pi^2} \frac{\partial^2}{\partial x_i \partial x_k} \frac{s_1\left(\frac{\mu r}{b}\right) - s_1\left(\frac{\mu r}{a}\right)}{r}.$$

Since for $p = q$ the functions

$$\gamma_{ii}(p,p;\mu) = (18\pi^2)^{-1}(a^{-3} + 2b^{-3})\mu^2,$$

it follows from (7), by Lemma 1.1 of paper [2], that the diagonal elements of the spectral matrix for $p = q$ satisfy the relation

$$\lim_{|x|\to\infty} |x|^{-2} \operatorname*{Var}_{x}^{x+1/h} \{\vartheta_{ii}(p,p;\mu)\} < \frac{25}{\pi^2 h}(a^{-3} + 2b^{-3}), \qquad i = 1,2,3,$$

uniformly for all $p \in \Omega_h$.

If $p \neq q$, then, applying the Cauchy-Bunyakovsky inequality, we obtain that all elements of the spectral matrix satisfy the relation

$$\lim_{|x|\to\infty} |x|^{-2} \operatorname*{Var}_{x}^{x+1/h} \{\vartheta_{ik}(p,q;\mu)\} < \frac{25}{\pi^2 h}(a^{-3} + 2b^{-3}), \qquad i,k = 1,2,3, \tag{8}$$

uniformly in p and q from the closed domain Ω_h.

From (7) and (8), and from the remark to Lemma 2.1 of paper [2], the assertion of the theorem follows, if as the family of kernels $T(N, \lambda)$ appearing in Lemma 2.1 of paper [2] one takes kernels of the form

$$T(N, \lambda) = \begin{cases} 1, & \text{if } |\lambda| < N, \\ \frac{1}{2}, & \text{if } |\lambda| = N, \\ 0, & \text{if } |\lambda| > N. \end{cases}$$

Remark. The matrix $\|\gamma_{ik}(p, q; \mu)\|_{i,k=1}^{3}$ is the spectral matrix of the operator $\mathcal{L}$ in the entire space $\dot{E}_3$.

Received
26 II 1957

CITED LITERATURE

[1] B. M. Levitan, *Matem. sborn.*, **35** (77), 2 (1954).
[2] V. A. Marchenko, *Izv. Akad. Nauk SSSR, ser. matem.*, **19**, No. 6 (1955).
[3] A. Pleijel, *Ark. f. Mat., Astr. och Fys.*, **27 A**, No. 13, 74 (1940).
[4] T. Bourouladze, *Soobshch. AN GruzSSR*, **15**, No. 4 (1954).
[5] A. Love, *Mathematical Theory of Elasticity*, Moscow-Leningrad, 1935, p. 317.

Note: Figure translations are in progress. See original paper for figures.