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Abstract

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ON THE ORDER OF APPROXIMATION OF FUNCTIONS BY LINEAR POSITIVE OPERATORS

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Let $\mathcal{L}_n(f;x)$ be a linear positive operator whose value, for any continuous function, is an algebraic or trigonometric polynomial of degree not exceeding n. Put

$$\Delta_n = \mathcal{L}_n(f;x) - f(x). \tag{1}$$

E. V. Voronovskaya (^{1}), studying the order of approximation of functions by S. N. Bernstein polynomials, established that the order of smallness of the quantity Δ_n in this case is, generally speaking, no higher than $1/n$, and proved the equality

$$\lim_{n\to\infty} n\Delta_n = \frac{1}{2}x(1-x)f''(x). \tag{2}$$

In this note it will be shown that the slow order of convergence to zero of the quantity Δ_n occurs for all linear and positive polynomial operators. In addition, an analogue of equality (2) will be established for arbitrary sequences of positive operators.

Theorem 1. If $\{\mathcal{L}_n(f;x)\}$ is a sequence of linear positive operators such that the value of the operator $\mathcal{L}_n(f;x)$ for any function continuous on the interval $-1 \le x \le 1$ is an algebraic polynomial of degree not exceeding n, then at least one of the sequences

$$\{n^2\|\mathcal{L}_n(1;x)-1\|\}, \qquad \{n^2\|\mathcal{L}_n(t;x)-x\|\}, \qquad \{n^2\|\mathcal{L}_n(t^2;x)-x^2\|\}$$

does not tend to zero.

Proof. We have

$$\begin{aligned} |\mathcal{L}_n(|t|;x) - |x|| &\le |\mathcal{L}_n(|t|;x) - |x|\,\mathcal{L}_n(1;x)| + |x|\,|\mathcal{L}_n(1;x)-1| \\ &\le |\mathcal{L}_n(|t|-|x|;x)| + |\mathcal{L}_n(1;x)-1|\,. \end{aligned} \tag{3}$$

Using the properties of linear positive operators (monotonicity and the Cauchy-Bunyakovsky inequality), we obtain

$$\begin{aligned}|\mathcal{L}_n(|t|-|x|;x)| &\leq \mathcal{L}_n(|t-x|;x) \leq \\ &\leq \sqrt{\mathcal{L}_n\{(t-x)^2;x\}\,\mathcal{L}_n(1;x)} = \\ &= \sqrt{\{\mathcal{L}_n(t^2;x)-x^2\}-2x\{\mathcal{L}_n(t;x)-x\}+x^2\{\mathcal{L}_n(1;x)-1\}}\,\sqrt{\mathcal{L}_n(1;x)}.\end{aligned} \tag{4}$$

Since $\mathcal{L}_n(f;x)$ is a polynomial operator, it follows that

$$E_n(|x|) \leq \|\mathcal{L}_n(|t|;x)-|x|\|. \tag{5}$$

From (3), (4), and (5) we obtain

$$nE_n(|x|) \leq n^2\|\mathcal{L}_n(1;x)-1\|+ \tag{6}$$

$$+\sqrt{\|\mathcal{L}_n(1;x)\|}\sqrt{n^2(\|\mathcal{L}_n(t^2;x)-x^2\|+2\|\mathcal{L}_n(t;x)-x\|+\|\mathcal{L}_n(1;x)-1\|)}.$$

If the theorem were not true, then the right-hand side of the last inequality would tend to zero. But this is impossible, since $nE_n(|x|)$ does not tend to zero.

It follows from the theorem proved that the order of approximation of functions by linear positive and polynomial operators is not higher than $1/n^2$, even for analytic functions. A similar assertion is easily carried over also to the trigonometric case. In what follows, sequences of linear positive and polynomial operators will be indicated for which the order of approximation $1/n^2$ holds for any twice differentiable function, $\|f''(x)\| < \infty$.

Theorem 2. Let $\varphi_n(x)$, $n = 1, 2, ...$, be nondecreasing functions on the interval $a \leq x \leq b$, and

$$\Phi_n(f) = \int_a^b f(x)\,d\varphi_n(x).$$

Let $\psi(x)$ be a continuous function on this interval,

$$\psi(c) = 0, \qquad \psi(x) > 0, \qquad x \neq c, \qquad a \leq c \leq b.$$

In order that from the equality

$$\lim_{x\to c}\frac{f(x)}{\psi(x)} = A < \infty \tag{7}$$

where $f(x)$ is any function for which the last limit exists, there follow the equality

$$\lim_{n\to\infty} \frac{\Phi_n(f)}{\Phi_n(\psi)} = A, \tag{8}$$

it is necessary and sufficient that for every $\delta > 0$ the equality

$$\lim_{n\to\infty} \frac{\alpha_n(\delta)}{\Phi_n(\psi)} = 0 \tag{9}$$

hold, where

$$\alpha_n(\delta) = \int_{|x-c|\geq\delta} d\varphi_n(x). \tag{10}$$

Proof. Putting $f(x) = \psi^2(x)$, we obtain

$$\Phi_n(f) = \int_a^b f(x)\, d\varphi_n(x) \geq \int_{|x-c|\geq\delta} f(x)\, d\varphi_n(x) \geq m\alpha_n(\delta),$$

where m is the least value of the function $f(x)$ on the set $|x-c| \geq \delta$. Consequently,

$$\frac{\Phi_n(f)}{\Phi_n(\psi)} \geq m\frac{\alpha_n(\delta)}{\Phi_n(\psi)}. \tag{11}$$

If (9) does not hold for some $\delta > 0$, then the right-hand side, and hence the left-hand side, of the last inequality do not tend to zero, although

$$\lim_{x\to c} \frac{f(x)}{\psi(x)} = \lim_{x\to c} \psi(x) = 0.$$

To prove the sufficiency of the conditions of the theorem, note that from (7) there follows the inequality

$$|f(x) - A\psi(x)| < \varepsilon\psi(x), \qquad |x-c| < \delta. \tag{12}$$

Now we have

$$\Phi_n(f) - A\Phi_n(\psi) = \int_a^b \{f(x) - A\psi(x)\}\, d\varphi_n(x) =$$

$$= \int_{|x-c|\leqslant\delta} \{f(x) - A\psi(x)\}\, d\varphi_n(x) + \int_{|x-c|\geqslant\delta} \{f(x) - A\psi(x)\}\, d\varphi_n(x).$$

But

$$\int_{|x-c|\geqslant\delta} |f(x) - A\psi(x)|\, d\varphi_n(x) \leqslant M\alpha_n(\delta), \qquad M = \|f(x) - A\psi(x)\|,$$

$$\int_{|x-c|<\delta} |f(x) - A\psi(x)|\, d\varphi_n(x) < \varepsilon \int_{|x-c|<\delta} \psi(x)\, d\varphi_n(x) \leqslant \varepsilon\Phi_n(\psi).$$

Consequently,

$$|\Phi_n(f) - A\Phi_n(\psi)| < \varepsilon\Phi_n(\psi) + M\alpha_n(\delta).$$

From this inequality, (9), and the arbitrariness of $\varepsilon > 0$, the theorem follows.

To obtain Voronovskaya’ s theorem from this theorem, it is necessary to note that

$$B_n(1;x) = 1, \qquad B_n\{(t-x);x\} = 0, \qquad B_n\{(t-x)^2;x\} = \frac{x(1-x)}{n},$$

$$\alpha_n(\delta) = \sum_{|k/n-x|\geqslant\delta} C_n^k x^k (1-x)^{n-k} = o\left(\frac{1}{n}\right).$$

Since

$$\alpha_n(\delta) = o(B_n\{(t-x)^2;x\}), \qquad 0 < x < 1,$$

by Theorem 2, from the equality

$$\lim_{t\to x} \frac{f(t) - f(x) - (t-x)f'(x)}{(t-x)^2} = \frac{1}{2}f''(x)$$

there follows the equality

$$\lim_{n\to\infty} \frac{B_n(f;x) - f(x)B_n(1;x) - f'(x)B_n(t-x;x)}{B_n\{(t-x)^2;x\}} =$$

$$= \lim_{n\to\infty} \frac{B_n(f;x) - f(x)}{\dfrac{x(1-x)}{n}} = \frac{1}{2} f''(x),$$

i.e. Voronovskaya’ s theorem.

Let us also apply Theorem 2 to Jackson’ s operators

$$D_n(f;x) = \frac{3}{2n(2n^2+1)} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) \left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^4 dt.$$

We have

$$\alpha_n(\delta) = \frac{3}{2n(2n^2+1)} \frac{1}{\pi} \left\{ \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^4 dt \right\} = O\left(\frac{1}{n^3}\right).$$

On the other hand,

$$D_n\left(\sin^2 \frac{t-x}{2}; x\right) = \frac{3}{2n(2n^2+1)} \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos nt\right) \frac{\sin^2(nt/2)}{\sin^2(t/2)} dt =$$

$$= \frac{3}{2(2n^2+1)} = O\left(\frac{1}{n^2}\right).$$

Applying Theorem 2, from the equality

$$\lim_{t\to x} \frac{f(t) - f(x) - f'(x)\sin(t-x)}{\sin^2 \frac{t-x}{2}} = 2f''(x)$$

we obtain

$$\lim_{n\to\infty} \frac{D_n(f;x) - f(x)}{D_n\left(\sin^2 \frac{t-x}{2}; x\right)} = \lim_{n\to\infty} \frac{4}{3} n^2 \{D_n(f;x) - f(x)\} = 2f''(x).$$

Consequently,

$$\lim_{n\to\infty} 4n^2 (D_n(f;x) - f(x)) = 6f''(x).$$

It would not be difficult to show that the last relation holds uniformly for all x, if the periodic function $f(x)$ has a continuous second derivative. Since the linear and positive operator $D_n(f,x)$ is a trigonometric polynomial of order not exceeding $2n$, the remark to Theorem 1 concerning the existence of operators realizing the order of approximation $1/n^2$ for good functions has also been proved.

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REFERENCES

1. E. V. Voronovskaya, DAN, A, 79 (1932).

Note: Figure translations are in progress. See original paper for figures.

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