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Abstract

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DIFFUSION OF RADIATION IN A SEMI-INFINITE MEDIUM

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PHYSICS

The problem of the diffusion of radiation in a semi-infinite medium arises in astrophysics (in the study of stellar and planetary atmospheres) and in geophysics (in the study of water basins). Usually the various special cases of this problem are considered separately from one another. In this note we shall show that the solutions of all problems on the diffusion of radiation in a semi-infinite medium, differing in the arrangement of the radiation sources, are expressed by means of one and the same function, depending only on the optical depth.

We shall assume that isotropic scattering of radiation occurs in the medium, and that the probability of survival of a quantum in an elementary act of scattering is equal to λ. The determination of the radiation field in the given medium is reduced to solving the following integral equation for the function $B(\tau)$:

$$B(\tau) = \frac{\lambda}{2} \int_0^\infty B(\tau') \ \mathrm{Ei}\,|\tau - \tau'|\, d\tau' + g(\tau), \tag{1}$$

where the function $g(\tau)$ characterizes the arrangement of the radiation sources. If the function $B(\tau)$ has been found, then the intensities of the radiation traveling at optical depth τ at an angle ϑ to the normal are expressed by the formulas

$$I(\tau, \vartheta) = \int_\tau^\infty B(\tau') e^{-(\tau' - \tau)\sec\vartheta} \sec\vartheta \, d\tau' \quad \left(\vartheta < \frac{\pi}{2}\right); \tag{2}$$

$$I(\tau, \vartheta) = -\int_0^\tau B(\tau') e^{-(\tau' - \tau)\sec\vartheta} \sec\vartheta \, d\tau' \quad \left(\vartheta > \frac{\pi}{2}\right). \tag{3}$$

The formal solution of equation (1) has the form

$$B(\tau) = g(\tau) + \int_0^\infty \Gamma(\tau, \tau') g(\tau') \, d\tau', \tag{4}$$

where $\Gamma(\tau, \tau')$ is the resolvent. Taking into account the probabilistic meaning of the resolvent, one can obtain ((1), p. 225) the following equation for its determination:

$$\frac{\partial \Gamma}{\partial \tau} + \frac{\partial \Gamma}{\partial \tau'} = \Gamma(\tau, 0)\Gamma(0, \tau'), \tag{5}$$

which gives (for $\tau' > \tau$):

$$\Gamma(\tau, \tau') = \Phi(\tau' - \tau) + \int_0^\tau \Phi(x)\Phi(x + \tau' - \tau)\, dx, \tag{6}$$

where it is denoted that $\Gamma(\tau, 0) = \Phi(\tau)$. As for the function $\Phi(\tau)$, it is determined by the integral equation

$$\Phi(\tau) = K(\tau) + \int_0^\tau K(\tau - \tau')\Phi(\tau')\, d\tau', \tag{7}$$

where

$$K(\tau) = \frac{\lambda}{2} \int_0^1 e^{-\tau/\eta} \varphi(\eta)\, \frac{d\eta}{\eta}, \tag{8}$$

and $\varphi(\eta)$ is the function of V. A. Ambartsumian, defined by him by the equation (2)

$$\varphi(\eta) = 1 + \frac{\lambda}{2}\eta\varphi(\eta) \int_0^1 \varphi(\zeta) \frac{\rho\zeta}{\eta + \zeta}. \tag{9}$$

Applying the Laplace transform to (7), we obtain

$$\int_0^\infty \Phi(\tau)e^{-m\tau}\, d\tau = \frac{1}{1 - \frac{\lambda}{2} \int_0^1 \varphi(\eta) \frac{d\eta}{1 + m\eta}} - 1. \tag{10}$$

From (10), for $m = \frac{1}{\zeta} \geqslant 1$, incidentally, it follows that

$$\int_0^\infty \Phi(\tau)e^{-\tau/\zeta}\, d\tau = \varphi(\zeta) - 1. \tag{11}$$

Inversion of the Laplace transform leads to the following asymptotic formula for the function $\Phi(\tau)$ for large τ:

$$\Phi(\tau) = \frac{e^{-k\tau}}{\frac{\lambda}{2}\int_0^1 \varphi(\eta)\frac{\eta\, d\eta}{(1-k\eta)^2}}, \tag{12}$$

where k is connected with λ by the relation

$$\frac{\lambda}{2k}\lg\frac{1+k}{1-k} = 1. \tag{13}$$

From (10) one can also obtain exact and approximate formulas for $\Phi(\tau)$ (we shall dwell on this later).

Thus, the determination of the radiation field in a semi-infinite medium for arbitrary radiation sources is reduced to finding the function $\Phi(\tau)$. If this function is known, then formula (6) gives the resolvent $\Gamma(\tau,\tau')$, formula (4) the function $B(\tau)$, and formulas (2) and (3) the radiation intensity $I(\tau,\vartheta)$.

Let us consider some particular cases of the problem of radiation diffusion in a semi-infinite medium.

1) Let

$$g(\tau) = Ge^{-m\tau}, \tag{14}$$

where G and m are constants. Using (4), (5), and (10), we find

$$B(\tau, m) = \frac{G}{1-\frac{\lambda}{2}\int_0^1 \varphi(\eta)\frac{d\eta}{1+m\eta}}\left[e^{-m\tau} + \int_0^\tau \Phi(\tau')e^{-m(\tau-\tau')}\, d\tau'\right]. \tag{15}$$

Formula (15) is valid not only for positive m, but also for negative ones, if $-m < k$.

If the medium is illuminated by parallel rays incident at an angle $\arccos\zeta$ to the normal and producing, on a surface perpendicular to them, an illumination equal to πS, then $g(\tau) = \frac{\lambda}{4}Se^{-\tau/\zeta}$. Setting in (15) $G = \frac{\lambda}{4}S$ and $m = \frac{1}{\zeta} \geqslant 1$, we find

$$B\left(\tau, \frac{1}{\zeta}\right) = \frac{\lambda}{4}S\varphi(\zeta)\left[e^{-\tau/\zeta} + \int_0^\tau \Phi(\tau')e^{-(\tau-\tau')/\zeta}\, d\tau'\right] \tag{16}$$

in agreement with the result obtained earlier ((1), p. 112).

2) Let

$$g(\tau) = \tau^n, \tag{17}$$

where n is a positive integer. Using (4) and (5), we obtain the recurrence relation

$$B_n(\tau) = n \int_0^\tau B_{n-1}(\tau')\, d\tau' + \Psi(\tau) \int_0^\infty \Phi(\tau')\, \tau'^n\, d\tau', \tag{18}$$

where

$$\Psi(\tau) = 1 + \int_0^\tau \Phi(\tau)\, d\tau. \tag{19}$$

The last integral in (18), by means of (7) and (8), is easily expressed in terms of the moments of the function $\varphi(\eta)$, equal to

$$\alpha_n = \int_0^1 \varphi(\eta)\eta^n\, d\eta. \tag{20}$$

For a uniform distribution of radiation sources in the medium (i.e. for $g(\tau) = 1$) we have

$$B_0(\tau) = \frac{\Psi(\tau)}{\sqrt{1-\lambda}}, \tag{21}$$

for $g(\tau) = \tau$

$$B_1(\tau) = \frac{1}{\sqrt{1-\lambda}} \int_0^\tau \Psi(\tau')\, d\tau' + \frac{\lambda}{2} \frac{\alpha_1}{1-\lambda} \Psi(\tau) \tag{22}$$

and so on.

3) Suppose that pure scattering of radiation takes place in the medium ($\lambda = 1$) and that the radiation sources are located at infinitely great depth. Denoting the function $B(\tau)$ for this case by $B_*(\tau)$ and taking into account the probabilistic meaning of the function $\Gamma(\tau, \tau')$, we obtain: $B_*(\tau) \sim \Gamma(\tau, \infty)$, or, using (6),

$$B_*(\tau) = B_*(0)\Psi(\tau). \tag{23}$$

Formula (23) gives, in particular, the solution of the problem of radiative equilibrium of a stellar photosphere. Denoting by πF the radiation flux in the photosphere (independent of τ), we have

$$F = 2\int_0^\infty \mathrm{Ei}_2\, \tau B_*(\tau)\, d\tau = 2B_*(0)\int_0^\infty \mathrm{Ei}_2\, \tau\Psi(\tau)\, d\tau. \tag{24}$$

Substituting here (19) and (11), we obtain the well-known relation

$$F = 2B_*(0)\alpha_1 = \frac{4}{\sqrt{3}}B_*(0). \tag{25}$$

In the theory of photospheres the function $B_*(\tau)$ is usually represented in the form

$$B_*(\tau) = \frac{3}{4}F[\tau + q(\tau)]. \tag{26}$$

Comparing (23) and (26), we arrive at the following relation between the functions $q(\tau)$ and $\Psi(\tau)$:

$$q(\tau) = \frac{\Psi(\tau)}{\sqrt{3}} - \tau. \tag{27}$$

Thus, in all the problems considered above the function $B(\tau)$ is expressed very simply in terms of the function $\Phi(\tau)$ or $\Psi(\tau)$.

If we know the solution of any one problem (exact or approximate), then, finding from it an expression for the function $\Phi(\tau)$, we can, by means of the formulas given, also obtain the corresponding solutions of the other problems.

Let us take, as an example, the approximate expression for the function $B_*(\tau)$ obtained by Chandrasekhar ([3]):

$$B_*(\tau) \simeq \frac{3}{4}F\left(\tau + \frac{1}{\sqrt{3}}\right). \tag{28}$$

It is easy to see that in the adopted approximation

$$\Phi(\tau) \simeq \sqrt{3}, \qquad \varphi(\eta) \simeq 1 + \eta\sqrt{3}. \tag{29}$$

Substituting (29) into (16), we find the approximate solution of the problem of radiation diffusion in a medium illuminated by parallel rays (for $\lambda = 1$):

$$B\left(\tau, \frac{1}{\zeta}\right) \simeq \frac{S}{4}(1 + \zeta\sqrt{3})\left[e^{-\tau/\zeta} + \sqrt{3}(1 - e^{-\tau/\zeta})\zeta\right]. \tag{30}$$

The results obtained in this note can easily be generalized to the case of a medium of finite optical thickness.

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REFERENCES

[1] V. V. Sobolev, *Transfer of Radiant Energy in the Atmospheres of Stars and Planets*, Moscow, 1956.
[2] V. A. Ambartsumian, *DAN*, 38, No. 8 (1943).
[3] S. Chandrasekhar, *Radiative Transfer*, IL, 1953, p. 83.

Note: Figure translations are in progress. See original paper for figures.