MATHEMATICS

M. I. VISHIK and Corresponding Member of the Academy of Sciences of the USSR L. A. LYUSTERNIK

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Abstract

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M. I. VISHIK and Corresponding Member of the Academy of Sciences of the USSR L. A. LYUSTERNIK

ON SOME ELLIPTIC EQUATIONS OF EVEN ORDER CONTAINING SMALL PARAMETERS AT THE HIGHEST DERIVATIVES AND DEGENERATING INTO EQUATIONS OF FIRST (AND, IN GENERAL, ODD) ORDER

In paper (^{1}) the asymptotics of the behavior of the solution of the first boundary-value problem for elliptic equations (of even order, containing parameters at the highest derivatives), which degenerates (for $\varepsilon = 0$) into the solution of the first boundary-value problem for an elliptic equation of a lower even order, was investigated. As will be shown, this method for studying asymptotics carries over, with the corresponding complications, also to the case when the degenerate equation is an equation of odd order.

1. **The Dirichlet problem for an equation of second order degenerating into the Cauchy problem for an equation of first order** (^{2}, ^{3}). Since the passage to the n-dimensional case presents no difficulties, in this section we shall confine ourselves to the plane case. Consider the elliptic equation

$$L_\varepsilon u_\varepsilon \equiv \varepsilon L_2 u_\varepsilon + L_1 u_\varepsilon = h, \tag{1}$$

$$L_2 u_\varepsilon \equiv a(x,y)\,\partial^2 u/\partial x^2 + 2b\,\partial^2 u/\partial x\,\partial y + c\,\partial^2 u/\partial y^2 \quad (a > 0),$$

$$L_1 u \equiv \partial u/\partial x - du, \qquad d > c^2 > 0$$

(in view of (1), by means of a change of variables a broad class of first-order differential operators can be reduced to this form).

Let Q be a domain bounded by a curve Γ, such that only two characteristics of L_1: $y = y_0$ and $y = y_1 > y_0$, touch Γ at the points $A(x_0, y_0)$, $B(x_1, y_1)$ and there have orders of contact, respectively, $k-1$, $l-1$ ($k, l \geq 2$). These points divide Γ into arcs Γ^- and Γ^+, consisting, respectively, of points of entry and exit of the characteristics. We shall say that the parameters of the problem have smoothness p if the coefficients a, b, c, d, h have continuous p-th derivatives in $\overline{Q}$, and the curve Γ is continuously differentiable $p+1$ times. We solve equation (1) under the condition

$$u_\varepsilon \big|_{\Gamma} = 0. \tag{2}$$

We shall also consider the Cauchy problem

$$L_1 w = h, \qquad w \big|_{\Gamma^+} = 0. \tag{3}$$

Let us note that in a neighborhood of the point A the derivatives $\partial w/\partial y$, $\partial^2 w/\partial y^2$ are unbounded: $\partial w/\partial y = O((y - y_0)^{1/k-1})$, $\partial^2 w/\partial y^2 = O((y - y_0)^{1/k-2})$, and analogously in a neighborhood of the point B.

Assuming that the parameters of the problem have smoothness $2n$, we construct two recurrent processes. The first consists in constructing the functions $w_0 = w, w_1, \dots, w_{n-1}$, where for $i > 0$

$$L_1 w_i = -L_2 w_{i-1}, \qquad w_i \big|_{\Gamma^+} = 0. \tag{4}$$

Then, if $\bar{w}_n = w_0 + \varepsilon w_1 + \dots + \varepsilon^{n-1} w_{n-1}$, then

$$L_\varepsilon \bar{w}_n = h + \varepsilon^n L_2 w_{n-1}. \tag{4'}$$

The second recurrent process, closely connected with the first, serves for the construction of "boundary layers" compensating the discrepancies in the boundary conditions on Γ^- of the solutions u_ε and w_i, and is a transfer to our problem of the process described in §§ 2, 3 of (1). Suppose that the parameters of the problem have smoothness $2n+2$. We introduce in a neighborhood of Γ^- a coordinate system (r, φ), namely: we construct a system of "transversals," i.e. vectors $\overline{PR}$ of length $\eta > 0$, drawn from points P of the arc Γ^- into Q, with smoothness of order $2n$; moreover, for a sufficiently small choice of η the transversals do not intersect; the coordinate r of a point S of the transversal $\overline{PR}$ is its distance PS, and φ is the length of the part AP of the arc Γ^-. In the new coordinates

$$L_\varepsilon u = \varepsilon\,(\alpha\,\partial^2 u/\partial r^2 + 2\beta\,\partial^2 u/\partial r\,\partial\varphi + \gamma\,\partial^2 u/\partial\varphi^2 + \dots) + \delta\,\partial u/\partial r + \chi\,\partial u/\partial\varphi - du;$$

here

$$\alpha(r, \varphi) = \alpha(\varphi) + r\beta_1(r, \varphi), \qquad \delta(r, \varphi) = \delta(\varphi) + r\delta_1(r, \varphi), \qquad \delta(\varphi) > 0.$$

We give the second splitting of the operator L_ε:

$$L_\varepsilon u \equiv M_\varepsilon u + R_\varepsilon u, \qquad M_\varepsilon u \equiv \varepsilon\alpha(\varphi)\,\partial^2 u/\partial r^2 + \delta(\varphi)\,\partial u/\partial r. \tag{5}$$

We construct a sequence of functions $v_i^{(0)} + v_i^{(1)}$ $(i = 0, 1, \dots, n-1)$, where

$$M_\varepsilon v_0^{(0)} = 0, \qquad v_0^{(0)}\big|_{r=0} = -w\big|_{r=0}, \qquad v_0^{(0)} \approx 0 \quad \text{for } r/\varepsilon \gg 1$$

(the last condition proves that $v_0^{(0)}$ has the character of a boundary layer). Hence

$$v_0^{(0)} = -w(\varphi, 0)\exp(-\omega r/\varepsilon), \qquad \omega = \omega(\varphi) = \delta(\varphi)/\alpha(\varphi) > 0.$$

Next we construct $v_0^{(1)}$ as the solution of the following problem:

$$M_\varepsilon v_0^{(1)} = -R_\varepsilon v_0^{(0)}, \qquad v_0^{(1)}\big|_{r=0} = 0, \qquad v_0^{(0)} \approx 0 \quad \text{for } r/\varepsilon \gg 1.$$

Similarly,

$$M_\varepsilon v_i^{(0)} = 0, \qquad v_i^{(0)}\big|_{r=0} = -w_i\big|_{r=0}, \qquad v_i^{(0)} \approx 0 \quad \text{for } r/\varepsilon \gg 1;$$

$$\varepsilon M_\varepsilon v_i^{(1)} = -(\varepsilon R_\varepsilon v_i^{(0)} + R_\varepsilon v_{i-1}^{(1)}), \qquad v_i^{(1)}\big|_{r=0} = 0, \qquad v_i^{(1)} \approx 0 \quad \text{for } r/\varepsilon \gg 1.$$

It is easy to see that $R_\varepsilon v_i^{(1)} = O(\varepsilon)$ $(i = 0, \dots, n-1)$, and, consequently, putting

$$\overline{v}_{n,\varepsilon} = v_0^{(0)} + v_0^{(1)} + \varepsilon(v_1^{(0)} + v_1^{(1)}) + \dots + \varepsilon^{n-1}(v_{n-1}^{(0)} + v_{n-1}^{(1)}),$$

we obtain outside neighborhoods of the points A and B:

$$L_\varepsilon \overline{v}_{n,\varepsilon} = \varepsilon^{n-1} R_\varepsilon v_{n-1}^{(1)} = O(\varepsilon^n). \tag{6}$$

Multiplying the function $\overline{v}_{n,\varepsilon}$ by a "smoothing" factor, i.e. a factor equal to 1 in a neighborhood of Γ^-, equal to 0 at some distance from Γ^-, and sufficiently smooth, we obtain a smooth function, which we shall still denote by $\overline{v}_{n,\varepsilon}$, and which we extend by zero to the whole remaining part of Q. Then outside neighborhoods of the points A and B we shall have, as before,

$$L_\varepsilon \overline{v}_{n,\varepsilon} = O(\varepsilon^n). \tag{7}$$

Denoting $z_{n,\varepsilon} = u_\varepsilon - \overline{w}_n - \overline{v}_{n,\varepsilon}$, we have from (1), (4′), and (7), outside neighborhoods of the points A and B:

$$L_\varepsilon z_{n,\varepsilon} = O(\varepsilon^n). \tag{8}$$

Theorem 1. If the parameters of the problem have smoothness of order 3, then

$$u_\varepsilon = w + v_0^{(0)} + v_0^{(1)} + z_{1,\varepsilon},$$

where w is the solution of problem (3); $v_0^{(0)}$ and $v_0^{(1)}$ are functions of boundary-layer type; $z_{1,\varepsilon}$ tends uniformly to 0 on Q as $\varepsilon \to 0$, and throughout the whole domain Q

$$|z_{1,\varepsilon}| \leq \min C\left((y-y_0)^{1/k},\ \varepsilon(y-y_0)^{k-1},\ \varepsilon(y_1-y)^{1/l-1},\ (y_1-y)^{1/l}\right). \tag{9}$$

Thus, outside any fixed neighborhood of the points A and B, $|z_{1,\varepsilon}| = O(\varepsilon)$.

Let us now consider the Hilbert space of functions defined on Q, where

$$(f,\varphi) = \iint_Q f\cdot\varphi\,dx\,dy.$$

Theorem 2. If $h \in \mathcal{L}_2$ and the coefficients a, b, c have bounded first derivatives, then u_ε converges weakly in $\mathcal{L}_2$ to w^*, and the "weak asymptotics"

$$(u_\varepsilon,\psi) - (w,\psi) = O(\sqrt{\varepsilon})$$

holds for every differentiable function ψ that vanishes in neighborhoods of A and B.

The proof is based on consideration of the properties of solutions of the equation adjoint to (3), and on constructing for this equation an analogue of the boundary layer $v_0^{(0)}$.

Theorem 3. If the parameters of the problem have smoothness of order 1, $k<3$, $l<3$, then for the residual $z = u_\varepsilon - w - v_0^{(0)}$ the following estimates hold:

$$\|z\|^2 \leq C\varepsilon, \qquad \|\partial z/\partial x\|^2 + \|\partial z/\partial y\|^2 \leq C. \tag{10}$$

Since $\|\partial v_0^{(0)}/\partial r\|^2 = O(\varepsilon^{-1})$, the second of inequalities (10) indicates that the principal part of the boundary layer has already been removed from the residual z.

Theorem 4. If the parameters of the problem have smoothness of order $2n+k+2$, then in $Q - U(A) - U(B)$ the following estimates hold for the residual $z_{n,\varepsilon} = u_\varepsilon - w_n - v_{n,\varepsilon}$:

$$\|z_{n,\varepsilon}\| = O(\varepsilon^n), \qquad \|D_1 z_{n,\varepsilon}\| = O(\varepsilon^{n-1/2}), \qquad \|D_i z_{n,\varepsilon}\| = O(\varepsilon^{n-i+1}), \tag{11}$$

$2 \leq i \leq k+2$ (D_i denotes any i-th derivative).

If, however, one considers $z_{n,\varepsilon}$ in Q', where $\overline{Q'} \subset Q$, then

$$\|D_i z_{n,\varepsilon}\| = O(\varepsilon^{n-1}) \qquad (k+2 \geq i \geq 2). \tag{11'}$$

The combination of these theorems with the results and methods of papers (4,5) makes it possible to obtain analogous asymptotic estimates for the solutions $p_t(x,y) = p(x,y,t)$ of the parabolic equation

$$\frac{\partial}{\partial t} p_t - L_\varepsilon p_t = h, \qquad p_t|_\Gamma = 0 \tag{12}$$

(for arbitrary initial values from $\mathcal{L}_2$), where $\varepsilon = \varepsilon(t)$ tends to 0 as $t \to \infty$.

2. Higher-order equations. We shall call an odd-order partial differential equation **one-characteristic** if its order is $2k+1$ and if at each point it has $2k$ complex characteristics and one real characteristic. We shall restrict ourselves to the case of an equation which, by a change of variables, can be reduced to the form

$$L_{2k+1} w \equiv \frac{\partial}{\partial x_1} M_{2k} w(x_1, \dots, x_m) + M'_{2k} w = h, \tag{13}$$

where M_{2k}, under the homogeneous conditions of the first boundary-value problem, is a positive elliptic operator of order $2k$; M'_{2k} is an operator of order $2k$. We regard this equation as one obtained as a result of the degeneration, as $\varepsilon \to 0$, of the positive elliptic equation

$$L_\varepsilon u_\varepsilon \equiv \sum_{s=2k+1}^{2(k+1)} \varepsilon^{s-(2k+1)} L_s u_\varepsilon = h, \tag{14}$$

where L_s are operators of order s. Equation (14) is considered in a domain Q with boundary Γ under the boundary conditions

$$\partial^i u_\varepsilon / \partial n^i \big|_\Gamma = 0 \qquad (i = 0, 1, \dots, k+l-1). \tag{15}$$

It turns out that, under certain conditions, the solution u_ε of problem (14), (15) tends to the solution w of equation (13) under the boundary conditions

$$\partial^i w / \partial n^i \big|_\Gamma = 0 \quad (i = 0, 1, \dots, k-1); \qquad \partial^k w / \partial n^k \big|_{\Gamma^-} = 0, \tag{16}$$

where Γ^- is the part of the boundary Γ consisting of the points of entry of the characteristics. We shall therefore call this problem the first boundary-value

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* We note that the weak convergence of u_ε to w was also independently proved by I. Kopachek and O. A. Ladyzhenskaya.

problem for equation (13). For an equation of first order this problem coincides with the Cauchy problem.

The two recurrent processes connected with the construction of the functions w_i and of the boundary layers v_i are defined as in the preceding case and as in $^{(1)}$.

For the solvability of problem (14), (15) for small ε it is enough to require the fulfillment of condition II (inequality (5) of $^{(1)}$) and the positivity of the operator L_{2k+1}:

$$(L_{2k+1}u, u) \geq c^2\|u\|^2.$$

The construction of the boundary layer is carried out as above and as in $^{(1)}$, on the basis of the splitting (7), (8) of the work $^{(1)}$ (with, in formulas (8), (9), the replacement of $\sum_{j=2k}^{2(k+l)}$ by $\sum_{j=2k+1}^{2(k+l)}$).

Lemma 1 of $^{(1)}$ is modified in the sense that the characteristic equation occurring in it has l roots with negative real part on Γ^+ and $l-1$ such roots on Γ^-. This means that, in constructing the boundary layer by a formula analogous to (11) of $^{(1)}$, one obtains as many degrees of freedom as there are conditions which, in our case, the function $v_0^{(0)}(\rho, \varphi, \varepsilon)$ must satisfy in order that, for example, the function $w + v_0^{(0)}$ satisfy the lost boundary conditions (15). We formulate one of the results (an analogue of Theorem 3) which pertains to the case $k = 0$, occurring in applications $^{(6)}$.

Theorem 5. *Under the corresponding smoothness conditions on the parameters of the problem, for the residual*

$$z_{n,\varepsilon} = u_\varepsilon - \bar{w}_n - \bar{v}_{n,\varepsilon},$$

where u_ε *is the solution of problem (14), (15) for* $k = 0$,

$$\bar{w}_n = w + \varepsilon w_1 + \cdots + \varepsilon^{n-1} w_{n-1}, \qquad \bar{v}_{n,\varepsilon} = v_0 + \varepsilon v_1 + \cdots + \varepsilon^{n-1} v_{n-1},$$

and these functions are obtained as a result of both iteration processes, in any subdomain Q' *not containing a neighborhood of the set of points of tangency of the characteristics with the boundary of the domain, the estimates*

$$\varepsilon^{2l-1}\|D_l z_{n,\varepsilon}\|^2 + \cdots + \varepsilon\|D_1 z_{n,\varepsilon}\|^2 + \|z_{n,\varepsilon}\|^2 \leq C\varepsilon^{2n}$$

hold.

Thus, already for $n = 1$ the boundary layer is captured. More precise estimates, analogous to (11), (11′), also hold.

Remark 1. Lemma 1 of $^{(1)}$ also generalizes to the cases of degeneration: from one-characteristic equations into elliptic equations, and from one-characteristic equations into one-characteristic equations. For different boundary points, the characteristic equation $P_\varphi(\lambda) = 0$ has exactly as many roots with negative real part as the number of boundary conditions for the nondegenerate equation that is lost at this point for the degenerate one.

Remark 2. Analogous phenomena (connected with the presence of a boundary layer) are observed for the asymptotics of the difference of the eigenfunctions of the degenerate and nondegenerate operators under homogeneous boundary conditions, for example, for the first boundary-value problem.

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Note: Figure translations are in progress. See original paper for figures.

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