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Abstract

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MATHEMATICS

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COVERING OF AREAS FOR FUNCTIONS REGULAR IN AN ANNULUS

(Presented by Academician V. I. Smirnov, 22 XI 1956)

A function $w = f(z) \in \mathfrak{M}$ if in the annulus $1 < |z| < R$ $w = f(z)$ is regular and

$$\frac{1}{2\pi i}\int_L \frac{f'(z)}{f(z)}\,dz \geqslant 1,$$

where L is a contour not homologous to zero.

The question of covering areas for functions of the class $\mathfrak{M}$ was posed by G. Ya. Khazaliya in the papers ([1,2]). In paper ([1]) the problem of minimizing the area of the image obtained under the mapping of the annulus $1 < |z| < R$ by a function $w = f(z) \in \mathfrak{M}$ is solved. In paper ([2]) the notion of a star of the Riemann surface onto which the annulus $1 < |z| < R$ is mapped by a function $w = f(z) \in \mathfrak{M}$ is introduced, and its area S^* is estimated: $S^* \geqslant \pi(R^2 - 1)$. Unfortunately, the definition of a star has shortcomings, which, in particular, were pointed out in ([3]); moreover, the existence of a star was not proved. We give a somewhat different definition of a star (cf. ([2,4])).

1. Let $\mathfrak{R}$ and $\mathfrak{R}_{\tau,\sigma}$ be Riemann surfaces onto which, respectively, the annuli $1 < |z| < R$ and $1+\tau < |z| < R-\sigma$, $\tau, \sigma > 0$, are mapped by the function $w = f(z) < \mathfrak{M}$; let $\Gamma_\tau, \Gamma_\sigma$ be the images of the circles $|z| = 1 + \tau$ and $|z| = R - \sigma$, obtained under this mapping. Let two arbitrary sequences of numbers $\tau_n \to 0$ and $\sigma_n \to 0$, $n \to \infty$, be given, satisfying the conditions: $\tau_k > \tau_{k+1}$, $\sigma_k > \sigma_{k+1}$, $k = 1, 2, \ldots$. Let an arbitrary system of rays L_k, $k = 1, 2, \ldots, m$, drawn from the point $w = 0$, be given.

2. We cut the Riemann surface $\mathfrak{R}_{\tau_1,\sigma_1}$ over all its sheets:

a) by the system of rays L_k, $k = 1, 2, \ldots, m$;

b) by the system of rays drawn from the point $w = 0$ through all branch points of $\overline{\mathfrak{R}_{\tau_1,\sigma_1}}$;

c) by the system of all rays issuing from the point $w = 0$ and touching the boundary of the Riemann surface $\mathfrak{R}_{\tau_1,\sigma_1}$.

The same rays cut each of the Riemann surfaces $\mathfrak{R}_{\tau_2,\sigma_2}$, $\mathfrak{R}_{\tau_3,\sigma_3}$, ... into a certain totality of parts. Let $\omega_1, \omega_2, \ldots, \omega_s$ be the sectors formed by neighboring rays

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of the indicated subdivision, and let their apertures be equal, respectively, to $\alpha_1, \alpha_2, \dots, \alpha_s$. Among the parts of the Riemann surfaces obtained under the indicated subdivision and contained in the sector ω_q, $q = 1, 2, \dots, s$, we single out all possible sequences of them (let their number be k_q*) of the form

$$\mathfrak{R}^{q,j}_{\tau_1,\sigma_1} \subset \mathfrak{R}^{q,j}_{\tau_2,\sigma_2} \subset \mathfrak{R}^{q,j}_{\tau_3,\sigma_3} \subset \dots, \quad j = 1, 2, \dots, k_q \ (k_q \geqslant 1), \quad 1 \leqslant q \leqslant s,$$

where $\mathfrak{R}^{q,j}_{\tau_k,\sigma_k}$ is such a part of the Riemann surface $\mathfrak{R}_{\tau_k,\sigma_k}$, $k = 1, 2, \dots$, lying in the sector ω_q, that the increment of $\arg w$ along all pieces $\Gamma_{\tau_k}, \Gamma_{\sigma_k}$ entering the boundary of $\mathfrak{R}^{q,j}_{\tau_k,\sigma_k}$, when they are traversed in the positive direction, is respectively equal to $-\alpha_q$ and $+\alpha_q$.

3. On the Riemann surface $\mathfrak{R}_{\tau_2,\sigma_2}$, $1 \leqslant q \leqslant s$, $1 \leqslant j \leqslant k_q$, mark:

a) all branch points of $\mathfrak{R}^{q,j}_{\tau_2,\sigma_2}$, whose projection onto the w-plane lies outside the projection of $\mathfrak{R}^{q,j}_{\tau_1,\sigma_1}$;

* It is clear that the number k_q of sequences depends on the structure of the Riemann surface.

b) all those points of the boundary of the Riemann surface $\mathfrak{R}^{q,j}_{\tau_2,\sigma_2}$ which have projection onto the w-plane outside the projection of $\mathfrak{R}^{q,j}_{\tau_1,\sigma_1}$, at which the rays drawn from the point $w = 0$ touch the boundary of $\mathfrak{R}^{q,j}_{\tau_2,\sigma_2}$.

From the point $w = 0$, through the marked points, draw a system of rays. Cut the Riemann surface $\mathfrak{R}^{q,j}_{\tau_2,\sigma_2}$ along all its sheets in the following way: draw a cut, not passing along $\mathfrak{R}^{q,j}_{\tau_1,\sigma_1}$, from each of the marked points along the corresponding ray to $w = \infty$ or $w = 0$. Among the parts of the Riemann surface obtained under such a partition, choose the one which contains $\mathfrak{R}^{q,j}_{\tau_1,\sigma_1}$; denote it by $V^{q,j}_{\tau_2,\sigma_2}$. On the Riemann surface $\mathfrak{R}^{q,j}_{\tau_3,\sigma_3}$, by means of $V^{q,j}_{\tau_2,\sigma_2}$, mark all points of the same type as the points marked by us on $\mathfrak{R}^{q,j}_{\tau_2,\sigma_2}$ by means of $\mathfrak{R}^{q,j}_{\tau_1,\sigma_1}$. Add to them the points marked by us earlier on $\mathfrak{R}^{q,j}_{\tau_2,\sigma_2}$. Separate from the surface $\mathfrak{R}^{q,j}_{\tau_3,\sigma_3}$, by means of $V^{q,j}_{\tau_2,\sigma_2}$, the part $V^{q,j}_{\tau_3,\sigma_3}$ by partitioning it along the rays drawn from the point $w = 0$ through the resulting set of points, just as we separated $V^{q,j}_{\tau_2,\sigma_2}$ from $\mathfrak{R}^{q,j}_{\tau_2,\sigma_2}$ by means of $\mathfrak{R}^{q,j}_{\tau_1,\sigma_1}$.

Continuing in the same way the partition of the Riemann surfaces $\mathfrak{R}^{q,j}_{\tau_4,\sigma_4}$, $\mathfrak{R}^{q,j}_{\tau_5,\sigma_5}$, ..., we obtain a sequence of their parts:

$$V^{q,j}_{\tau_1,\sigma_1} = \mathfrak{R}^{q,j}_{\tau_1,\sigma_1} \subset V^{q,j}_{\tau_2,\sigma_2} \subset V^{q,j}_{\tau_3,\sigma_3} \subset \dots .$$

In this case the parts $V^{q,j}_{\tau_k,\sigma_k}$, $k = 1, 2, \dots$, are simply connected; every ray drawn from the point $w = 0$ through a point of $V^{q,j}_{\tau_k,\sigma_k}$ has only one segment which is a cross-section of $V^{q,j}_{\tau_k,\sigma_k}$; the increment of $\arg w$ along all arcs Γ_{τ_k}, Γ_{σ_k} entering into the boundary of $V^{q,j}_{\tau_k,\sigma_k}$, when they are traversed in the positive direction, is respectively equal to $-\alpha_q$ and $+\alpha_q$.

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4. Let j_q $(q = 1, 2, \dots, s)$ be some arbitrary number among the numbers $1, 2, \dots, k_q$. Denote by $V_n(j_1, \dots, j_s)$ the totality of the domains

$$\sum_{q=1}^{s} V_{\tau_n,\sigma_n}^{q,j_q}, \qquad n = 1, 2, \dots .$$

The **star** $\mathfrak{R}^*$ of the Riemann surface $\mathfrak{R}$ with respect to the system L_k, $k = 1, 2, \dots, m$, of rays drawn from the point $w = 0$, is the open set situated on the Riemann surface $\mathfrak{R}$ and which is the limiting set for some sequence $V_n(j_1, j_2, \dots, j_s)$ as $n \to \infty$ (it is clear that if at least one of the numbers $k_1, k_2, \dots, k_s$ is greater than one, then the star is not unique).

It is evident that the star $\mathfrak{R}^*$ is simply connected and that a ray drawn from the point $w = 0$ through a point of $\mathfrak{R}^*$ has only one segment, lying in $\mathfrak{R}^*$, with endpoints on the boundary of $\mathfrak{R}^*$.

The star defined in the indicated manner exists. Using the method of A. F. Bermant ([5]), p. 192, one can obtain a result more general than the result of G. Ya. Khazalia:

$$\left(1 + \frac{S}{\pi}\right)\left(1 + \frac{s}{\pi}\right) \geq R^4,$$

where S is the area of the star, s is the area of its transform; equality is possible only for the function $f(z) = \varepsilon z$, $|\varepsilon| = 1$.

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20 XI 1956

CITED LITERATURE

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[4] A. F. Bermant, *Matem. sborn.*, **20**, 55 (1947).
[5] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, 1952.

Note: Figure translations are in progress. See original paper for figures.

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