Mathematics

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1957

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Abstract

Full Text

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ON THE CONVERGENCE OF A MODIFICATION OF THE GALERKIN METHOD

(Presented by Academician M. V. Keldysh, February 7, 1957)

N. I. Pol' skii ((^{1})) considered the question of convergence of the generalized Galerkin method for linear equations of the form

[Au + B( )u = f]

in a certain Hilbert space, assuming (A) to be a positive definite self-adjoint operator. In the present note it is shown that convergence of one modification of the Galerkin method will also hold under less stringent restrictions imposed on the operator (A).

1. Consider, on the interval (0 < x < 1), the equation

$$L[y] \equiv y^{(n)} + \sum_{j=0}^{n-1} p_j(x,\lambda) y^{(j)} = f(x), \tag{1}$$

where (n) is an arbitrary natural number, with the boundary conditions

$$U_i[y] \equiv \sum_{k=0}^{n-1} \left[\alpha_{ik} y^{(k)}(0) + \beta_{ik} y^{(k)}(1)\right] = 0 \quad (i = 1, 2, \dots, n). \tag{2}$$

We shall regard the functions (p_k(x,)) as continuous on the segment ([0,1]) for every ( ) from a domain (D) of the complex plane and as analytic functions of ( ) in (D). The function (f(x)) is assumed continuous. In addition, we assume, as usual, that the matrix composed of the constants (*{ik}) and (* {ik}) has rank (n). The boundary-value problem thus obtained will be called problem ((L)).

Let

$$A[y] \equiv y^{(n)} + \sum_{j=0}^{n-1} q_j(x) y^{(j)}, \tag{3}$$

where the operator (A) is chosen so that the problem

[A[y]=0,   U_i[y]=0   (i=1,2,…,n)]

has no nontrivial solution.

Choose a system of linearly independent functions

$$\varphi_1(x),\ \varphi_2(x), \dots, \varphi_k(x), \dots, \tag{4}$$

satisfying the boundary conditions (2) and such that the system of functions

$$\psi_1(x),\ \psi_2(x), \dots, \psi_k(x), \dots, \tag{5}$$

where

$$\psi_k = A[\varphi_k], \tag{6}$$

forms a complete orthonormal system in (L_2(0,1)).
We seek an approximate solution of problem (L) in the form

$$y_N = \sum_{k=1}^{N} \eta_k^{(N)} \varphi_k, \tag{7}$$

where the constants (_k^{(N)}) are found from the system

$$(L[y_N] - f, \psi_i) = 0 \quad (i = 1, 2, \dots, N). \tag{8}$$

Then the following fundamental theorem holds.

Theorem 1. *The functions (y_N) and their derivatives of order less than (n) converge uniformly to the solution and to the corresponding derivatives of the solution of problem (L), and the sequence ({y_N^{(n)}}) converges in mean to the (n)-th derivative of the solution. The eigenvalues of problem (L) are the limits of the eigenvalues of the systems (8).*

The proof, as in the work of M. V. Keldysh ((^{2})), is based on the theory of infinite systems of linear algebraic equations.

For the practical application of the method the following theorem is useful.

Theorem 2. *The functions (_k) may always be chosen to be polynomials.*

If problem (L) has a unique solution and the system ({L[_i]}) is complete, then Theorem 1 immediately implies the convergence of the least-squares method for problem (L).

2. We now consider, in a domain (Q) of (n)-dimensional space, the elliptic equation

$$L[u] \equiv \sum_{i,k=1}^{n} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = f \tag{9}$$

with coefficients (b_i) and (c) depending linearly on the parameter ( ), and some homogeneous boundary condition under which a Green' s function exists for equation (9). For simplicity we shall assume

$$u = 0 \tag{10}$$

on the boundary ( ) of the domain (Q). The coefficients of equation (9) will be assumed continuous and to have a sufficiently large number of partial derivatives with respect to (x_i), and the boundary of the domain to have continuous curvature. Let

$$A[u] \equiv \sum_{i,k=1}^{n} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b'_i \frac{\partial u}{\partial x_i} + c' u, \tag{11}$$

where the operator (A) is such that the problem

[A[u]=0,   u|_{ }=0]

has no nontrivial solution.

As in item 1, choose a system of linearly independent functions

$$\varphi_1, \varphi_2, \dots, \varphi_k, \dots, \tag{12}$$

satisfying the boundary condition (10) and such that the system of functions

$$\psi_1, \psi_2, \dots, \psi_k, \dots, \tag{13}$$

where

$$\psi_k = A[\varphi_k], \tag{14}$$

is a complete orthonormal system in (L_2(Q)).

We shall seek an approximate solution of the problem in the form

$$u_N = \sum_{k=1}^{N} \eta_k^{(N)} \varphi_k, \tag{15}$$

where the coefficients (_k^{(N)}) are determined from the system

$$(L[u_N] - f, \psi_i) = 0 \qquad (i = 1, 2, \dots, N). \tag{16}$$

Then the following theorem holds.

Theorem 3. *The functions (u_N) and their first derivatives converge in the mean to the solution and to the corresponding derivatives of the solution of the problem under consideration, and the eigenvalues of this problem are obtained by passing to the limit from the eigenvalues of system (16).*

From this theorem, in particular, there follows the convergence of the least-squares method, if the problem has a unique solution.

3. Finally, let us consider in a Hilbert space (H) an equation of the form

$$Lu \equiv Au + B(\lambda)u = f, \tag{17}$$

where (A) and (B( )) are linear operators in (H), and the norm of the operator (B( )) depends analytically on ( ) in some domain (D) of the complex plane.

Suppose that the operators (A^{-1}) and (B( )A^{-1}) are completely continuous and that the domain of definition of the operator (A) belongs to the domain of definition of the operator (B( )). We seek approximate solutions of equation (17) in the form

$$u_N = \sum_{k=1}^{N} \eta_k^{(N)} \varphi_k, \tag{18}$$

where (_k) belong to the domain of definition of the operator (A), which is dense in (H), and the system ({ _k}), where (_k=A _k), is complete in (H). The coefficients (_k^{(N)}) are determined from the system

$$(Lu_N - f, \psi_i) = 0 \qquad (i = 1, 2, \dots, N). \tag{19}$$

Then the following theorem holds.

Theorem 4. *The sequence of approximate solutions (u_N) converges in (H) to the solution of equation (17), and the eigenvalues of equation (17) are the limits of the "approximate" eigenvalues.*

From this theorem, and under its conditions, there follows the convergence of the least-squares method for equations of the form (17), if the equation has a unique solution and the system ({L _k}) is complete in (H).

Received
13 XI 1956

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References

1. N. I. Pol' skii, *Ukr. Mat. Zhurn.*, 7, No. 1 (1955).

2. M. V. Keldysh, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 6, No. 6 (1942).

Note: Figure translations are in progress. See original paper for figures.