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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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M. E. TEMCHENKO

ON THE STABILITY OF ONE OF THE POSITIONS OF DYNAMIC EQUILIBRIUM OF A MECHANICAL SYSTEM

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In paper ([1]) it was shown that an axisymmetric rigid body suspended on an absolutely flexible, inertia-free string and rotating with constant angular velocity ω has, in addition to the stationary motion in which the axis of symmetry of the rotating body is vertical, also stationary motions in which the axis of dynamic symmetry of the body is inclined to the vertical. In ([2]), on the basis of theoretical investigations, it was shown that the vertical position of dynamic equilibrium of such a body is stable for any parameters characterizing its motion.

Below a theoretical investigation is given of the stability of one of the above-mentioned stationary motions of a rigid body, in which the axis of dynamic symmetry of the body is inclined to the vertical.

Fig. 1

1°. Let us introduce a fixed coordinate system $\xi\eta\zeta$, the origin of which we place at the point O_1 (Fig. 1). We direct the ζ-axis of this system vertically upward, and place the axes ξ and η in the horizontal plane. At the center of gravity of the body—the point C—we place the origin of a coordinate system xyz rigidly connected with the body; we direct its z-axis along the axis of its dynamic symmetry, and place the axes x and y in the plane perpendicular to the z-axis.

We shall derive the equations of motion of the body under investigation in the form of Lagrange equations of the second kind. As generalized coordinates we choose the angles α, β, φ, ψ, and θ.

The angles α and β characterize the position of the string in space, the angle α characterizing the deviation of the string from the vertical, while the angle β is the angle formed by the projection of the string on the horizontal plane with the positive direction of the η-axis.

The angles φ, ψ, and θ characterize the position of the rigid body in space: the angle φ determines the deviation of the axis of the body' s proper rotation from the vertical; the angle ψ is the angle between the projection of the axis of the body' s proper rotation on the horizontal plane and the positive direction of the η-axis; and, finally, the angle θ is the angle of the body' s proper rotation.

The equations of motion in the case under consideration may be represented in the form

$$M\left[-l\cos\alpha\sin\beta\,\frac{d^2\xi_c}{dt^2}+l\cos\alpha\cos\beta\,\frac{d^2\eta_c}{dt^2}+l\sin\alpha\,\frac{d^2\zeta_c}{dt^2}\right]=-Mgl\sin\alpha,$$

$$M\left[-l\sin\alpha\cos\beta\,\frac{d^2\xi_c}{dt^2}-l\sin\alpha\sin\beta\,\frac{d^2\eta_c}{dt^2}\right]=0,$$

$$\begin{gathered}M\left[-a\cos\varphi\sin\psi\,\frac{d^2\xi_c}{dt^2}+a\cos\varphi\cos\psi\,\frac{d^2\eta_c}{dt^2}+a\sin\varphi\,\frac{d^2\zeta_c}{dt^2}\right]+\\+A\frac{d^2\varphi}{dt^2}-A\left(\frac{d\psi}{dt}\right)^2\sin\varphi\cos\varphi+C\left(\frac{d\theta}{dt}+\frac{d\psi}{dt}\cos\varphi\right)\frac{d\psi}{dt}\sin\varphi=-Mga\sin\varphi,\\M\left[-a\sin\varphi\cos\psi\,\frac{d^2\xi_c}{dt^2}-a\sin\varphi\sin\psi\,\frac{d^2\eta_c}{dt^2}\right]+A\frac{d}{dt}\left(\frac{d\psi}{dt}\sin^2\varphi\right)+\\+C\frac{d}{dt}\left[\left(\frac{d\theta}{dt}+\frac{d\psi}{dt}\cos\varphi\right)\cos\varphi\right]=0,\\C\frac{d}{dt}\left(\frac{d\theta}{dt}+\frac{d\psi}{dt}\cos\varphi\right)=0.\end{gathered}\tag{1}$$

Here M is the mass of the body under consideration; g is the acceleration of gravity; A and C are, respectively, the equatorial and axial moments of inertia of the body; l is the length of the string; a is the distance from the center of gravity of the body to the point at which it is attached to the string (Fig. 1); ξ_c, η_c, ζ_c are the coordinates of its center of gravity with respect to the coordinate system $\xi\eta\zeta$.

2°. It is not difficult to see that the equations of motion (1) admit the particular solution

$$\begin{gathered}\alpha=\alpha_0'=\text{const},\qquad\varphi=\varphi_0'=\text{const},\qquad\beta=\beta_0'+\omega t,\\\psi=\psi_0+\omega t,\qquad\theta=\theta_0'=\text{const}.\end{gathered}\tag{2}$$

Let us note here that the solution (2) is determined uniquely if the following ranges of possible values of the angles $\alpha_0, \beta_0, \varphi_0$ and ψ_0 are adopted:

$$0 \le \alpha_0' \le \frac{\pi}{2}, \qquad 0 \le \varphi_0 \le \frac{\pi}{2}, \qquad 0 \le \beta_0 \le 2\pi, \qquad 0 \le \psi_0 \le 2\pi. \tag{3}$$

By virtue of equations (1), between the constants $\alpha_0, \varphi_0, \beta_0$ and ψ_0 there hold either the relations

$$\begin{gathered} \beta_0 = \psi_0, \\ -Ml^2 \sin\alpha_0 \cos\alpha_0\, \omega^2 - Mal \cos\alpha_0 \sin\varphi_0\, \omega^2 = -Mgl \sin\alpha_0, \\ -Ma^2\omega^2 \sin\varphi_0 \cos\varphi_0 - Mal\omega^2 \sin\alpha_0 \cos\varphi_0 - A\omega^2 \sin\varphi_0 \cos\varphi_0 + \\ +C\omega^2 \cos\varphi_0 \sin\varphi_0 = -Mga \sin\varphi_0, \end{gathered} \tag{4}$$

or

$$\begin{gathered} \beta_0 = \psi_0 + \pi, \\ Ml^2 \sin\alpha_0 \cos\alpha_0\, \omega^2 - Mal\omega^2 \cos\alpha_0 \sin\varphi_0 = Mgl \sin\alpha_0, \\ -Ma^2 \sin\varphi_0 \cos\varphi_0\, \omega^2 + Mal \sin\alpha_0 \cos\varphi_0\, \omega^2 - A\omega^2 \sin\varphi_0 \cos\varphi_0 + C\omega^2 \cos\varphi_0 \sin\varphi_0 \\ = -Mga \sin\varphi_0. \end{gathered} \tag{5}$$

In what follows we shall restrict ourselves to those solutions (2) for which

$$0 < \alpha_0 < \frac{\pi}{2}, \qquad 0 < \varphi_0 < \frac{\pi}{2}. \tag{6}$$

The solution (2), when the relations (4) are satisfied, corresponds to the following motion of the mechanical system: the rigid body suspended on the string rotates with constant angular velocity ω about the vertical axis ζ. The angle between the direction of the string and the vertical is α, while the angle between the axis of the body' s own rotation and the vertical is φ. The projections of the string and of the axis of the body' s own rotation onto the horizontal plane form, with the positive direction of the η-axis, the angles β and ψ, respectively. Owing to the fulfillment of the first equality (4), the vertical axis ζ, the string l, and the axis of dynamic symmetry of the body lie in one plane.

A schematic representation of the position of dynamic equilibrium of the body considered is shown in (1) in Fig. 3 b.

The solution (2), when the relations (5) are satisfied, corresponds to the motion of a rigid body with constant angular velocity ω about the vertical axis ζ. The string l and the axis of symmetry of the body form with the vertical, respectively,

angles α, φ. As a consequence of the fact that, according to the first equality (5), $\beta_0 - \psi_0 = \pi$, the string l, the axis of proper rotation of the body, and the vertical also lie in one plane.

A schematic representation of the position of dynamic equilibrium considered here is given in ([1]) in Fig. 3 b.

3°. We shall take the solution (2) as unperturbed and investigate its stability with respect to part of the variables, namely with respect to the variables $\alpha, \beta, \dot{\alpha}, \dot{\beta}, \varphi, \dot{\psi}$ and $\dot{\theta}$, under the condition that between the angles $\alpha_0, \beta_0, \varphi_0$ and ψ_0 the dependence (4) holds. At the same time we shall keep in mind that, with respect to the variables β and ψ, as follows from the solution (2), the motion of the body under study will be unstable.

The investigation of stability in the present case will be carried out by means of Lyapunov' s second method ([3]). V. V. Rumyantsev* showed ([4]) that the Lyapunov theorems lying at the basis of his second method for investigating stability, with a certain change in formulation, are also valid for the case of stability with respect to part of the variables.

Thus, for example, in Lyapunov' s first theorem on the stability of motion ([3], p. 59), the function V should be understood as a positive definite function with respect to part of the variables. If, for example, the sign-constant function $V(x_1, x_2, \dots, x_n)$ does not depend explicitly on the time t, and the constant H can be chosen sufficiently small so that, under the conditions ([4])

$$t \geq t_0, \qquad \sum_{s=1}^{k} x_s^2 < H, \qquad x_{k+1}, \dots, x_n \text{ arbitrary} \tag{7}$$

the function $V(x_1, x_2, \dots, x_n)$ vanishes only when the variables x_s $(s = 1, 2, \dots, k;\ k < n)$ are all zero, then such a function will be called positive definite with respect to the variables $x_1, x_2, \dots, x_k$ ([4]). This theorem is used in proving the stability of the stationary motion under consideration.

In the present case we shall construct the Lyapunov function V, by the method of N. G. Chetaev ([5]), in the form of a linear combination of the first two integrals of the perturbed motion: the energy integral V_1 and the area integral V_2, namely

$$V = V_1 - \omega V_2, \tag{8}$$

where ω is a constant.

It is not difficult to show that the function V fully satisfies all the conditions of the theorem set forth in ([4]) on the stability of the motion of a mechanical system with respect to part of the variables.

Thus, in the case under consideration there is unconditional stability of the stationary motion of the body under consideration with respect to the variables $\alpha, \varphi, \dot{\alpha}, \dot{\beta}, \dot{\varphi}, \dot{\psi}$ and $\dot{\theta}$. With respect to the variables β and ψ, however, as was mentioned above, the motion is unstable.

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

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Note: Figure translations are in progress. See original paper for figures.