ON FINDING THE CLASS OF CONVERGENCE OF CERTAIN INTERPOLATION PROBLEMS

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Abstract

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MATHEMATICS

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ON FINDING THE CLASS OF CONVERGENCE OF CERTAIN INTERPOLATION PROBLEMS

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Let $\{L_n(F)\}$ be a system of linear functionals defined on the entire class of entire functions $F(z)$:

$$L_n(F) = \frac{1}{2\pi i}\int_C F(z)\, u_n\left(\frac{1}{z}\right)\frac{dz}{z}, \qquad u_n(z) = z^n + \sum_{k=1}^{\infty} a_{n,k} z^{n+k}.$$

By $\{P_n(z)\}$ we denote the system of polynomials biorthogonal to the system of functionals $\{L_n(F)\}$, i.e. satisfying the conditions

$$L_n(P_m) = \delta_{n,m}, \qquad n, m = 0, 1, 2, \ldots$$

We are interested in the problem of finding that class of entire functions in which

$$F(z) = \sum_{n=0}^{\infty} L_n(F) P_n(z).$$

We shall give a solution of this problem for the case when all $u_n(z/\lambda_n)$, starting with $n > n_0(z)$, do not vanish in the domain E and

$$\frac{u_{n+1}(z/\lambda_{n+1})}{u_n(z/\lambda_n)} \to u(z), \qquad u(0) = 0, \qquad u'(0) = 1, \tag{1}$$

uniformly in z inside E. At the same time, on the sequence λ_n we shall impose certain restrictions concerning its growth and very stringent restrictions concerning its smoothness. Namely, we shall assume that $\lambda_n = \lambda(n)$, where $\lambda(z)$ is regular in the entire z-plane with a cut $(-\infty, 0)$, and the following conditions are satisfied:

1. $\lambda(z) > 0$ for $z > 0$, and $w = \lambda(z)$ maps the half-plane $\operatorname{Re} z > 0$ onto a domain D bounded by curves which, in a neighborhood of $w = \infty$, have equations $\theta = \pm\frac{1}{2}\alpha(r)$, $w = re^{i\theta}$.

2. The function $\alpha(r)$ and its two derivatives are monotone for $r > r_1$. Conditions 1 and 2 are conditions on the smoothness of λ_n.

We shall consider the growth of λ_n to be restricted by the conditions:

$$3. \quad \frac{1}{2} < \rho \leq \infty, \qquad \rho = \lim_{r\to\infty} \frac{\pi}{\alpha(r)}; \qquad \lim_{r\to\infty} r\,\frac{\alpha'(r)}{\alpha(r)} = 0.$$

Conditions 3 mean that $\ln \lambda_n > \frac{1}{2}\ln n$, $\ln\ln\lambda_n = o(\ln n)$.

By $h(z)$ we shall denote that branch of the inverse function to $\lambda(z)$ which is positive for positive z.

When conditions 1-3 are satisfied for $\lambda(z)$, Lemmas 1 and 2 are valid.

Lemma 1. As $z \to \infty,\ z \in D\ (z = re^{i\theta})$,

$$z\frac{h'(z)}{h(z)} = \frac{\pi}{a(\ln r)} + O((\ln r)^{-2+\varepsilon});$$

$$\ln h(z) = \pi \int_{r_0}^{\ln r} \frac{dt}{a(t)} + C + \frac{\pi i\theta}{a(\ln r)} + O((\ln r)^{-1+\varepsilon}). \tag{2}$$

Lemma 2. The function $w = h_1(z)$, $h_1(z) = zh'(z)$, maps onto the half-plane $\operatorname{Re} w > 0$ the domain D_1, bounded by curves which, for $r > r_2$, have equations $\theta = \pm\frac{1}{2}\alpha_1(r)$ $(z = re^{i\theta})$. Moreover,

$$\frac{a(r)}{a_1(r)} = 1 - \frac{1}{\pi}\alpha'(r) + O(r^{-2+\varepsilon})$$

and

$$z\frac{h_1'(z)}{h_1(z)} = \frac{\pi}{a(\ln r)} - \alpha'(\ln r) + O((\ln r)^{-2+\varepsilon});$$

$$\ln h_1(z) = \pi \int_{r_0}^{\ln r} \frac{dt}{a(t)} + C_1 - \alpha(\ln r) + \frac{\pi i\theta}{a(\ln r)} + O((\ln r)^{-1+\varepsilon}). \tag{3}$$

Let $\Phi(z)$ denote the entire function defined by the equality

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{e^{h(\zeta)}}{\zeta - z}\,d\zeta, \tag{4}$$

where the contour L begins at infinity below the real axis (coinciding in a neighborhood of $\zeta = \infty$ with the curve $\vartheta = -\frac{1}{2}\alpha(r)$), and ends at infinity above the real axis (coinciding in a neighborhood of $\zeta = \infty$ with the curve $\theta = \frac{1}{2}\alpha(r)$). In the finite part of the plane the contour is drawn so that the point $\zeta = z$ remains to its left.

Lemma 3. As $z \to \infty$ outside the domain D,

$$\Phi(z) = O(1),$$

and as $z \to \infty$ inside D,

$$\Phi(z) = e^{h(z)} + O(1). \tag{5}$$

It should be noted that the extremely stringent smoothness requirements on λ_n were needed for the simplicity of constructing the function $\Phi(z)$, by means of which the desired class of entire functions is singled out. This function, by its meaning, must have the proper behavior at infinity. The severity of the smoothness conditions on λ_n does not lead to a narrowing of the class $\{L_n(F)\}$ for which the main result is proved, owing to the weak restrictive nature of condition (1), although it does make the characterization of this class less transparent.

Set

$$v_n(z) = \frac{1}{2\pi i}\int_C u_n\left(\frac{z}{\zeta}\right)\Phi(\zeta)\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int_{C_1} u_n\left(\frac{z}{\lambda_n t}\right)\Phi(\lambda_n t)\frac{dt}{t}. \tag{6}$$

Applying the saddle-point method to estimate the last integral, we obtain the basic estimate for $v_n(z)$.

Lemma 4. Let z lie in the domain E, in which relation (1) is valid, and suppose, in addition, that the point $t_n(z)$, defined by the equality

$$h_1(\lambda_n t_n(z)) = \varphi_n\left(\frac{z}{t_n(z)}\right), \qquad \varphi_n(\zeta) = \zeta\frac{u_n'(\zeta)}{u_n(\zeta)}, \tag{7}$$

lies (for $n > n_0(z)$) in the domain D. Then $v_n(z) \neq 0$ for $n > n_1(z)$ and

$$\lim_{n\to\infty}\frac{v_{n+1}(z)}{v_n(z)} = v(z), \tag{8}$$

where

$$v(z) = u(\xi(z))\exp\left\{\frac{1}{\rho}\varphi(\xi(z))\right\}, \tag{9}$$

$$\varphi(\zeta) = \zeta\frac{u'(\zeta)}{u(\zeta)}, \qquad \xi(z)[\varphi(\xi(z))]^{1/\rho} = \rho^{1/\rho}z, \qquad \xi(0) = 0, \qquad \rho = \lim_{r\to\infty}\frac{\pi}{a(r)}.$$

Lemma 5. In order that the point $t_n(z)$, for $n > n_0(z)$, lie in the domain D, it is necessary and sufficient that the point z lie in a simply connected domain containing $z = 0$, at all points of which

$$\operatorname{Re} z\frac{v'(z)}{v(z)} > 0. \tag{10}$$

Lemmas 4 and 5, in combination with certain general considerations set out in detail in (^{1}), and with Theorem 1 of the paper (^{2}), give the following final result:

Theorem 1. Let the function $v(z)$, defined by the equalities (8) and (9), be regular and univalent in a star-shaped domain $K \subset E$, mapped by the function $w = v(z)$ onto the disk. If the entire function $F(z)$ can be represented in the form

$$F(z) = \frac{1}{2\pi i}\int_C \Phi(z\zeta)f(\zeta)\,d\zeta,$$

where $f(\zeta)$ is regular outside K, and the contour C contains all singular points of $f(\zeta)$, then

$$F(z) = \sum_{n=0}^{\infty} L_n(F)P_n(z).$$

The latter series converges not only in every finite part of the plane, but also in the sense of the topology $\mathfrak{A}(\Phi, K)$ (^{1}).

In those cases when K is the maximal domain of univalence of $v(z)$, mapped by it onto the disk, Theorem 1 apparently cannot be improved.

Theorem 1 gives, in particular, for the Abel-Goncharov problem results in some respect (namely, with respect to the character of convergence of the interpolation series) even somewhat stronger than those set out in (^{1,3,4}). The application of Theorem 1 in this case is quite simple, since for the Abel-Goncharov problem

$$u_n(z) = \frac{z^n}{(z-\lambda_n)^{n+1}},$$

and, consequently,

$$u(z) = \frac{z}{z-1}.$$

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Note: Figure translations are in progress. See original paper for figures.

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