# ON THE THEORY OF OPERATIONAL CALCULUS

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## Abstract

## Full Text

## MATHEMATICS

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# ON THE THEORY OF OPERATIONAL CALCULUS

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In Heaviside’ s operational calculus one considers the operator $D = \frac{d}{dt}$. In the work (¹) of Mikusiński a rigorous foundation is given for Heaviside’ s operational calculus. Mikusiński constructs operators by extending the ring of functions, where convolution is taken as multiplication,

$$a(t) * b(t) = \int_0^t a(t-\xi)b(\xi)\,d\xi. \tag{1}$$

This method can be applied not only to the operator $\frac{d}{dt}$. In the present note an operational calculus is set out for the operator $\frac{d}{dt}\,t\,\frac{d}{dt}$. If, in Bessel’ s equation, the variable $x$ is replaced by the variable $t$ according to the formula $x = 2\sqrt{t}$, then the equation takes the form

$$\frac{d}{dt}\left(t\frac{dy}{dt}\right) + \left(1 - \frac{\nu^2}{4t}\right)y = 0. \tag{2}$$

Consequently, the operator $\frac{d}{dt}\,t\,\frac{d}{dt}$ is closely connected with the Bessel operator.

Let $C_2$ be the set of all twice differentiable functions defined on the half-line $0 \le t < \infty$, whose second derivatives are piecewise-continuous functions.

Define in $C_2$ multiplication by the formula

$$\varphi(t)\cdot\psi(t) = \frac{d}{dt}\left\{t\frac{d}{dt}\int_0^t d\xi \int_0^1 \varphi(x\xi)\psi[(1-x)(t-\xi)]\,dx\right\}. \tag{3}$$

Here $\varphi(t) \in C_2$, $\psi(t) \in C_2$. Addition in $C_2$ will be understood in the usual sense.

If instead of (1) one takes

$$a(t) * b(t) = \frac{d}{dt}\int_0^t a(t-\xi)b(\xi)\,d\xi, \tag{1'}$$

then the complete analogy between $(1')$ and (3) is evident. It should be noted that defining multiplication by formula $(1')$ somewhat simplifies Mikusiński' s theory, since one does not have to distinguish constants from constant functions. It is not difficult to verify that the multiplication defined in (3) again belongs to the set $C_2$, is commutative, associative, and has the distributive property with respect to addition. Moreover, if one of the factors in (3) is a number, for example if $\varphi(t) = \alpha$ is a constant, then

$$\alpha\cdot\psi(t) = \frac{d}{dt}\left\{t\frac{d}{dt}\int_0^t d\xi\int_0^1 \alpha\psi[(1-x)(t-\xi)]\,dx\right\} = \alpha\frac{d}{dt}\left\{t\int_0^1 \psi(xt)\,dx\right\} = \alpha\psi(t),$$

i.e., the product of a number by a function in the ring coincides with the ordinary product of a number by a function. If in (3) both factors are numbers, then the product in the sense of (3) coincides with the ordinary product of numbers. Let us prove that the ring thus constructed has no zero divisors.

Indeed, if $\varphi(t)\cdot\psi(t) = 0,\ 0 \leqslant t < \infty$, then from (3) it follows that

$$\int_0^t d\xi\int_0^1 \varphi(x\xi)\psi[(1-x)(t-\xi)]\,dx = c_1\ln t + c_2.$$

Since for $t = 0$ the left-hand side is equal to zero, $c_1 = c_2 = 0$. Consequently, for all $t \geqslant 0$,

$$\int_0^t d\xi\int_0^1 \varphi(x\xi)\psi[(1-x)(t-\xi)]\,dx = 0.$$

Replacing $t$ by $t\tau$ and making a change of variables in the integrals according to the formulas $\xi = ty,\ x = z/\tau$, we find

$$\int_0^t\int_0^\tau \varphi(yz)\psi[(t-y)(\tau-z)]\,dy\,dz = 0$$

for all $t \geqslant 0$ and $\tau \geqslant 0$. If $\psi(t)$ is not identically zero, then from the theorem of Mikusiński and Ryll-Nardzewski ($^2$) we obtain $\varphi(t) \equiv 0$. Consequently, the ring can be extended to a field of quotients. We denote the extended ring by $N(C_2)$. Following Mikusiński, we shall call the elements of the field operators. Denote the operator $1/t$ by the letter $B$. Consequently, the function $t$ corresponds to the operator $1/B$; therefore from (3) we find

$$\frac{1}{B}\psi(t) = \frac{d}{dt}\left\{ t\frac{d}{dt}\int_0^t d\xi \int_0^1 x\xi\psi[(1-x)(t-\xi)]\,dx \right\}.$$

Carrying out the calculations, we find

$$\frac{1}{B}\psi(t) = \int_0^t \frac{d\xi}{\xi}\int_0^\xi \psi(u)\,du. \tag{4}$$

It follows from this that if $\varphi(t) \in C_2$ and $\varphi(0) = 0$, then

$$B\varphi(t) = \frac{d}{dt}\left(t\frac{d\varphi}{dt}\right) \equiv t\varphi''(t) + \varphi'(t). \tag{5}$$

Thus, in the case $\varphi(t) \in C_2$ and $\varphi(0) = 0$, the product $B\varphi$ means applying to $\varphi$ the operator $\frac{d}{dt}\,t\,\frac{d}{dt}$.

From (4) it follows that

$$\frac{1}{B^n} = \frac{t^n}{(n!)^2},$$

whence

$$\frac{1}{B^{n+1}}\psi(t) = \frac{1}{(n!)^2}\int_0^t d\xi \int_0^1 \psi(x\xi)(1-x)^n(t-\xi)^n\,dx. \tag{6}$$

The equation

$$\frac{d}{dt}\left(t\frac{dy}{dt}\right) = ay \tag{7}$$

is satisfied by the functions $I_0(2\sqrt{at})$ and $K_0(2\sqrt{at})$. From (5) it follows that

$$B\left[I_0(2\sqrt{at}) - I_0(0)\right] = \frac{d}{dt}\left(t\frac{d}{dt}I_0(2\sqrt{at})\right)$$

or

$$B\left[I_0(2\sqrt{at}) - 1\right] = aI_0(2\sqrt{at}),$$

whence

$$\frac{B}{B-a} = I_0(2\sqrt{at}). \tag{8}$$

Using this formula, it is not difficult to obtain

$$\frac{B}{B+a} J_0(2\sqrt{at}), \qquad \frac{B}{B^2+a^2} = \operatorname{ber}(2\sqrt{at}), \qquad \frac{aB}{B^2+a^2} = \operatorname{bei}(2\sqrt{at}), \tag{9}$$

$$\frac{B^2}{B^2-a^2} = \frac{1}{2}\left[I_0(2\sqrt{at}) + J_0(2\sqrt{at})\right], \qquad \frac{aB}{B^2-a^2} = \frac{1}{2}\left[I_0(2\sqrt{at}) - J_0(2\sqrt{at})\right].$$

A large part of Mikusinski' s theory is applicable to the ring of operators $N(C_2)$, in particular the definition of the limit of a sequence of operators, the definition of the derivative of an operator function, series of operators, and integration of an operator function. Using this theory, by known methods one can considerably extend table (9). For example, it is not difficult, by differentiating with respect to the variable $a$, to obtain from (8)

$$\frac{B}{(B-a)^{n+1}} = \frac{1}{n!}\left(\frac{t}{a}\right)^{n/2} I_n(2\sqrt{at}). \tag{10}$$

The operational calculus for the operator $\dfrac{d}{dt}\, t\, \dfrac{d}{dt}$ can be constructed starting from the corresponding integral transform. The analogue of the Laplace transform here is the integral

$$\bar{\varphi}(p) = 2\int_0^\infty \varphi(t) K_0(2\sqrt{pt})\, dt. \tag{11}$$

If the measurable function $\varphi(t)$ satisfies the condition

$$|\varphi(t)| < Q e^{2\gamma_0\sqrt{t}}, \tag{12}$$

where $Q$ and $\gamma_0 > 0$ are constants, then the integral (11) converges in the domain $\operatorname{Re}\sqrt{p} > \gamma_0$ and represents an analytic function in this domain.

There is an inverse transform[3]

$$\varphi(t) = \frac{1}{2\pi i}\int_L \bar{\varphi}(p) I_0(2\sqrt{pt})\, dp, \tag{13}$$

where the path of integration $L$ is any parabola $\operatorname{Re}\sqrt{p} = \gamma > \gamma_0$.

Here $\varphi(t)$, in addition to condition (12), must satisfy the usual supplementary requirements, for example, have bounded variation in a neighborhood of each point of the line $0 < t < \infty$. At points of discontinuity of the function $\varphi(t)$, the integral (13) is equal to $\frac{1}{2}[\varphi(t+0)+\varphi(t-0)]$. For functions satisfying condition (12), the multiplication theorem is valid. If $\varphi(t)$ and $\psi(t)$ satisfy condition (12) and

$$\omega(t) = \int_0^t d\xi \int_0^1 \varphi(x\xi)\psi[(1-x)(t-\xi)]\,dx, \tag{14}$$

then there exists

$$\bar{\omega}(p) = 2\int_0^\infty \omega(t) I_0(2\sqrt{pt})\,dt \tag{15}$$

and

$$\bar{\omega}(p) = \bar{\varphi}(p)\bar{\psi}(p).$$

In conclusion, let us point out the connection between the table of values of the operators $F(B)$ and the table of values of the operators $F(D)$. If $F(B) = \varphi(t)$ and $F(D) = f(t)$, then

$$f(t) = \int_0^\infty \varphi(t\xi)e^{-\xi}\,d\xi;$$

here $\varphi(t)$ satisfies condition (12).

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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