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Abstract

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EXPANSION IN EIGENFUNCTIONS OF A SYSTEM OF SECOND-ORDER DIFFERENTIAL EQUATIONS

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The spectral expansion is studied for operators generated by a system of second-order differential expressions

$$l(y) = -y'' + P(x)y \qquad (0 \leq x < \infty), \tag{1}$$

in the case where the real symmetric matrix of order n, $P(x)$, is summable on the interval $(0, \infty)$. This note is a continuation of the author' s preceding note ([1]).

We introduce the following notation. Let L_0 denote the operator in the space $L_n^2(0, \infty)$ of vector-functions

$$y(x) = \{y_1(x), y_2(x), \dots, y_n(x)\},$$

summable with the square of the norm, generated by the differential expression (1) and the boundary conditions at zero

$$y'(0) = \theta y(0)$$

(θ is a Hermitian matrix). Denote by $\Omega_1(x, s)$, $\Omega_2(x, s)$ ($s^2 = \lambda$) linearly independent solutions of the matrix equation

$$l(Y) - \lambda Y = 0. \tag{2}$$

Further, denote by $\xi_1(s), \xi_2(s), \dots, \xi_n(s)$ the eigenvalues of the problem:

$$[A_1(s) - \xi A_2(s)]\rho = 0, \tag{3}$$

where

$$A_1(s) = \Omega_1'(0, s) - \theta\Omega_1(0, s),$$

$$A_2(s) = \Omega_2'(0, s) - \theta\Omega_2(0, s).$$

Finally, let

$$\Omega_i(x, s) = -\Omega_1(x, s)\xi_i(s) + \Omega_2(x, s), \qquad i = 1, 2, \dots, n.$$

Under the condition that the eigenvalue problem (3) has a complete system of orthogonal eigenvectors $\rho_1(s), \rho_2(s), \dots, \rho_n(s)$, the following theorems are valid:

Theorem 1 (Parseval equality). For every vector-function $f(x)$ belonging to $L_n^2(0, \infty)$, the equality

$$\int_0^\infty \|f(x)\|^2\,dx =$$

$$= \sum_{k=1}^\infty \frac{|a_k|^2}{\int_0^\infty \|y_k(x)\|^2\,dx} - \frac{1}{\pi}\int_0^\infty \frac{\sum_{i,j=1}^n F_i^*(s)\rho_i(s)\rho_j^*(s)\overline{c_i}c_jF_j(s)\,ds}{\sum_{i=1}^n (|\xi_i(s)|^2+1)\|\rho_i\|^2|c_i|^2},$$

where

$$\alpha_k = \int_0^\infty (f, y_k)\,dx;$$

y_k is an eigenfunction of L_θ; $c_1, c_2, \dots, c_n$ are constant numbers,

$$F_i(s) = \operatorname{l.\,i.\,m.}_{n\to\infty} \int_0^n \Omega_i(x, s) f(x)\,dx.$$

(The symbol l.i.m. denotes the limit in the sense of the norm in the Hilbert space generated by the spectral matrix of the operator L_θ. See, in this connection, ([3]).)

Theorem 2. For the kernel $K(x, \xi, \mu)$ of the resolvent of the operator L_θ the following integral representation holds ($\operatorname{Im}\mu \neq 0$):

$$K(x, \xi, \mu) = \sum_{k=1}^\infty \frac{y_k(x)y_k^*(\xi)}{(\lambda_k - \mu)\int_0^\infty \|y_k\|^2\,dx} - \frac{1}{\pi}\int_0^\infty \frac{\sum_{i,j=1}^n \Omega_i(x, s)\rho_i(s)\rho_j^*(s)c_i\overline{c_j}\Omega_j^*(\xi, s)}{(s^2 - \mu)\left[\sum_{i=1}^n \left(|\xi_i(s)|^2 + 1\right)\|\rho_i\|^2|c_i|^2\right]}\,ds.$$

The integral on the right-hand side of this equality converges absolutely and uniformly with respect to x, ξ in the region $0 \leq x, \xi < \infty$.

In proving these theorems, the method proposed by M. A. Naimark in his paper ([2]) is used. We shall give the principal points of the proof.

Linearly independent matrix solutions $\Omega_1(x, s)$, $\Omega_2(x, s)$ of equation (2) are constructed so that they satisfy the asymptotic formulas: as $x \to \infty$,

$$\Omega_1(x, s) = e^{isx}[1 + o(1)]$$

uniformly with respect to s, $|s| \geq r > 0$, $\operatorname{Im} s \geq 0$;

$$\Omega_2(x, s) = e^{-isx}[1 + o(1)]$$

uniformly with respect to s, $|s| \geq r > 0$, $\operatorname{Im} s \leq 0$; as $s \to \infty$,

$$\Omega_1(x, s) = e^{isx}[1 + O(1/s)], \quad \Omega_2(x, s) = e^{-isx}[1 + O(1/s)]$$

uniformly with respect to x, $0 \leq x < \infty$.

With the aid of these formulas we find asymptotic formulas for the eigenfunctions of the boundary-value problem on the finite interval $[0, b]$

$$l(y) - \lambda y = 0, \quad y'(0) - \theta y(0) = 0, \quad y(b) = 0. \tag{4}$$

The eigenvalues of this boundary-value problem form n infinite sequences $\{\lambda_k^{(i)}\}$, $i = 1, 2, \dots, n$, $k = 1, 2, 3, \dots$, such that

$$\lambda_k^{(i)} = [s_k^{(i)}]^2, \quad s_k^{(i)} = -\frac{k\pi}{b} - \frac{1}{2bi} \ln \xi_i \left(\frac{k\pi}{b}\right) + \frac{1}{b} o_j(1)$$

as $b \to \infty$.

Suppose that problem (3) has a complete system of eigenvectors $\rho_1(s), \rho_2(s), \dots, \rho_n(s)$; then the eigenfunctions of the boundary-value problem (4) are determined up to a constant vector $c = (c_1, c_2, \dots, c_n)$ and, as $b \to \infty$, satisfy the asymptotic formulas

$$y_k(x) = \sum_{i=1}^{n} \Omega_i(x, s_k)\rho_i(s_k)c_i + o(1), \quad k = 1, 2, 3, \dots .$$

Further, for the eigenfunction $y(x, s)$ of the boundary-value problem (4), corresponding to the eigenvalue $\lambda = s^2$, $s > 0$, as $b \to \infty$ the asymptotic formula

$$\frac{1}{b}\int_0^{\infty} \|y(x,s)\|^2\,dx = \sum_{i=1}^{n} \left(|\xi_i(s)|^2 + 1\right) \|\rho_i(s)\|^2\,|c_i|^2 + o(1) \qquad (5)$$

holds uniformly with respect to $s,\ 0 \le s < \infty$.

Further, the operator L_θ is self-adjoint, and its spectrum on the negative half-axis $\lambda < 0$ is discrete and bounded below, while on the positive half-axis $\lambda > 0$ it is continuous [1]. Taking this into account and using the asymptotic formulas for the eigenvalues and eigenfunctions of the boundary-value problem (4) and formula (5), from the corresponding spectral representations for the boundary-value problem (4), by passing to the limit as $b \to \infty$, we arrive at Theorems 1 and 2.

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REFERENCES

[1] Z. I. Biglov, DAN, **99**, No. 4 (1954).
[2] M. A. Naimark, Tr. Moscow Math. Soc., **3** (1954).
[3] I. Kats, Zap. Scientific-Research Institute of Mathematics and Mechanics and Kharkov Math. Soc., **22** (1950).

Note: Figure translations are in progress. See original paper for figures.