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Abstract

Full Text

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ON THE PROBLEM OF THE MOTION OF A HEAVY RIGID BODY WITH ONE FIXED POINT

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MECHANICS

1. Let $O\xi\eta\zeta$ be a fixed system of coordinate axes with the axis $O\zeta$ directed vertically upward; $Oxyz$ a moving system of coordinate axes directed along the principal axes of inertia of the body for its fixed point O. Suppose that among the principal moments of inertia of the body A, B, C no two are equal, and that the center of gravity of the body is located (for simplicity) on one of the principal axes of inertia; let the coordinates of the center of gravity be $x_0 > 0,\ y_0 = z_0 = 0$. The position of the rigid body in the space $O\xi\eta\zeta$ can be determined by three Euler angles $q_1 = \theta,\ q_2 = \varphi,\ q_3 = \psi$. The precession angle ψ is a cyclic coordinate, to which there corresponds the first integral of the equations of motion of a heavy rigid body with one fixed point in Lagrange form

$$\frac{\partial L}{\partial \psi'} = \left[(A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + C\cos^2\theta\right]\psi' +$$

$$+(A-B)\theta'\sin\theta\sin\varphi\cos\varphi + C\varphi'\cos\theta = n, \tag{1}$$

where $L = T + U$ is the Lagrange function; n is an arbitrary constant; the weight of the body will be denoted by P.

If we put $R = L - n\psi'$ and eliminate ψ' from the right-hand side of this equality by means of the integral (1), then we obtain the expression for the Routh function

$$R(q_1, q_2, q_1', q_2', n) = \frac{1}{2\left[(A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + C\cos^2\theta\right]}\Big\{[AB\sin^2\theta +$$

$$+(A\cos^2\varphi + B\sin^2\varphi)C\cos^2\theta]\theta'^2 - 2(A-B)C\theta'\varphi'\sin\theta\cos\theta\sin\varphi\cos\varphi +$$

$$+(A\sin^2\varphi + B\cos^2\varphi)C\sin^2\theta\,\varphi'^2+$$

$$+2(A-B)n\theta'\sin\theta\sin\varphi\cos\varphi + 2Cn\varphi'\cos\theta\} - Px_0\sin\theta\sin\varphi-$$

$$-\frac{1}{2}\frac{n^2}{(A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + C\cos^2\theta}. \tag{2}$$

In this case the Lagrange equations for the noncyclic coordinates will take the form

$$\frac{d}{dt}\left(\frac{\partial R}{\partial q_i'}\right) - \frac{\partial R}{\partial q_i} = 0 \quad (i = 1, 2); \tag{3}$$

after integration the coordinate ψ is determined by the quadrature

$$\psi = -\int \frac{\partial R}{\partial n}\,dt. \tag{4}$$

Since the function R does not depend explicitly on the time t, equations (3) have the first integral

$$H = \sum_{i=1}^{2} \frac{\partial R}{\partial q_i'} q_i' - R = h, \tag{5}$$

corresponding to the energy integral.

2. It is easy to see that equations (3), for certain values n_0 of the constant n, admit the particular solution $q_i = q_{i0}$ $(i = 1, 2)$, describing stationary motion of the body.

The constants q_{i0} $(i = 1, 2)$ are determined by the system of equations

$$\frac{\partial R}{\partial q_i} = 0 \qquad (i = 1, 2),$$

which can be written in the form

$$\begin{gathered}\left[Px_0\gamma - (A-C)\alpha\gamma\omega^2\right]\alpha - (B-C)\beta^2\gamma\omega^2 = 0,\\ Px_0\beta - (A-B)\alpha\beta\omega^2 = 0,\end{gathered} \tag{6}$$

where the following notation has been introduced:

$$\alpha = \sin\theta_0 \sin\varphi_0, \qquad \beta = \sin\theta_0 \cos\varphi_0, \qquad \gamma = \cos\theta_0, \qquad \omega = \frac{n_0}{A\alpha^2 + B\beta^2 + C\gamma^2}.$$

Obviously, equations (6) are satisfied if the constants satisfy the conditions

$$\alpha \neq 0, \qquad \beta \neq 0, \qquad \gamma = 0, \qquad \omega^2 = \frac{Px_0}{(A-B)\alpha},$$

or

$$\alpha \neq 0, \qquad \beta = 0, \qquad \gamma \neq 0, \qquad \omega^2 = \frac{Px_0}{(A-C)\alpha}. \tag{7}$$

The particular solutions (7) describe permanent rotations with constant angular velocities ω of a heavy rigid body about axes directed vertically upward and passing through the point O in the planes Oxy $(\gamma = 0)$ or Oxz $(\beta = 0)$.

Since for real motion $\omega^2 > 0$, the admissible axes in the indicated planes will be the half-lines for which $\alpha > 0$ in the cases $A > B,\ A > C$, or $\alpha < 0$ in the cases $B > A,\ C > A$.

The investigation of the stability of permanent rotations is of great significance for the problem of the motion of a rigid body with one fixed point. Necessary conditions for the stability of these rotations in the general case $x_0 \neq 0,\ y_0 \neq 0,\ z_0 \neq 0$ were obtained by Grammel (^{2}); with the aid of these conditions, Bottema (^{3}) attempted to consider the case $x_0 \neq 0,\ y_0 = z_0 = 0$, but made an error in the calculations. Sufficient conditions for stability of rotation about a principal axis of inertia in the Lagrange case were obtained by N. G. Chetaev (^{4}), and for the general and a number of special cases—in (^{5}).

We take the rotation about some permanent axis, situated, for example, in the plane Oxy and different from a principal axis of inertia $(\alpha, \beta \neq 0 \text{ or } 1,\ \gamma = 0)$, as the unperturbed one, assuming in the perturbed motion

$$\theta = \frac{\pi}{2} + x_1, \qquad \varphi = \varphi_c + x_2, \qquad n = n_0.$$

Substituting these values of the variables into function (2) and writing equations (3) with (6) taken into account, it is easy to see that the variational equations for the perturbed motion take the form

$$\begin{aligned} Dx_1'' + C\omega k x_2' - a x_1 = 0; \\ Cx_2'' - C\omega k x_1' - b x_2 = 0, \end{aligned} \tag{8}$$

where, for the constant coefficients, the notation has been introduced

$$
\begin{gathered}
a=(C-B)\omega^2, \qquad b=\frac{(A-B)\beta^2}{A\alpha^2+B\beta^2}\,[3(B-A)\alpha^2+B]\omega^2, \\
D=\frac{AB}{A\alpha^2+B\beta^2}, \qquad k=1+\frac{A-B}{C}\frac{B\beta^2-A\alpha^2}{A\alpha^2+B\beta^2}.
\end{gathered}
\tag{9}
$$

The equations (8) have the first integral

$$
H-H_0=\frac{1}{2}\left(D{x'_1}^2+C{x'_2}^2-ax_1^2-bx_2^2\right)=\text{const}, \tag{10}
$$

which is the first approximation of the integral of the form (5) for the full equations of the perturbed motion.

In the case when $a<0,\ b<0$, the function $H-H_0$ is sign-definite, and, according to Routh' s theorem $(^4)$, the unperturbed motion under consideration is stable with respect to the variables $q_i,\ q'_i\ (i=1,2)$, provided that the values of the constant n are not perturbed. However, the latter condition is inessential and may be omitted. Thus, permanent rotations about axes in the plane Oxy are stable with respect to the variables $\theta,\ \varphi,\ \theta',\ \varphi',\ \psi'$ $(^5)$ for all admissible axes in the case $B>C,\ B>A$, and also for those axes in the case $A>B>C$ for which

$$
\alpha^2>\frac{B}{3(A-B)}=\alpha_1^2. \tag{11}
$$

On the basis of equation (4), the instability of the permanent rotations with respect to the angle of precession ψ is obvious.

In the case when $a>0,\ b<0$ or $a<0,\ b>0$, the absence of a maximum of the modified force function is already revealed by the terms of the second order in its Taylor expansion, and, according to the converse of Lagrange' s theorem given by A. M. Lyapunov $(^6)$, the unperturbed motion is unstable under the action of potential forces; moreover, by Kelvin' s theorem $(^4)$, it cannot be stabilized by gyroscopic terms, since the degree of instability is equal to one.

On this basis we conclude the instability of permanent rotations about axes lying in the plane Oxy, in the case $C>B>A$ for all admissible axes, in the case $A>B,\ C>B$ when $\alpha^2>\alpha_1^2$, and in the case $A>B>C$ when $\alpha^2<\alpha_1^2$.

In the case $a>0,\ b>0$, corresponding to permanent rotations about axes for which $\alpha^2<\alpha_1^2$ when $A>B,\ C>B$, the degree of instability is equal to two and gyroscopic stabilization is possible.

If the unperturbed motion is stable, then, according to the fundamental theorem of N. G. Chetaev $(^4)$, the equations in variations have a sign-definite quadratic integral.

This integral for the case $a>0,\ b>0$ can be found in connection with the first integrals of equations (8). Indeed, it is easy to see that equations (8) also admit the first integral

$$\Gamma = 2\left(Dbx_1'x_2 - Cax_1x_2'\right) + C\omega k\left(ax_1^2+bx_2^2\right) +$$

$$+\frac{Db-Ca}{2C\omega k}\left(D{x'}_1{}^2 - C{x'}_2{}^2 + bx_2^2 - ax_1^2\right) = \text{const}, \tag{12}$$

analogous to the integral of N. G. Chetaev (⁷).

Consider the function

$$V = C\omega k(H-H_0)+\Gamma = \frac{(C\omega k)^2+Db-Ca}{2C\omega k}\,D{x'}_1{}^2 + 2Dbx_1'x_2 +$$

$$+\frac{(C\omega k)^2+Db-Ca}{2C\omega k}\,bx_2^2 + \frac{(C\omega k)^2-Db+Ca}{2C\omega k}\,C{x'}_2{}^2 - 2Cax_1x_2' +$$

$$+\frac{(C\omega k)^2-Db+Ca}{2C\omega k}\,ax_1^2. \tag{13}$$

For $a>0,\ b>0$, the conditions for the definiteness of the function (13) reduce to the single inequality

$$(C\omega k)^2 - Db - Ca - 2\sqrt{CDab} > 0, \tag{14}$$

which is satisfied.

We note that, for $a>0,\ b>0$, inequality (14) is also the condition that all roots of the characteristic equation be purely imaginary,

$$CD\sigma^4 - (Ca+Db-(C\omega k)^2)\sigma^2 + ab = 0. \tag{15}$$

Thus, for $a>0,\ b>0$, the conditions for gyroscopic stability of the permanent rotations are satisfied.

Whether such stability will actually occur depends on the continuability of the integral (12) of the variational equations to an integral for the complete equations of the perturbed motion. The solution of this question, still open, is of substantial importance for the problem of the motion of a heavy rigid body with one fixed point.

For permanent rotations about axes situated in the plane Oxz ($\beta = 0$), the results will coincide with those obtained if, in the latter, B is replaced by C and C by B.

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Note: Figure translations are in progress. See original paper for figures.

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