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Abstract

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ON A CERTAIN GENERALIZATION OF THE VARIATIONAL METHOD

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S. G. Mikhlin, in the work (^{1}) and others, gives a justification of the convergence of the variational method for the equation

$$Au = f, \qquad f \in H, \tag{1}$$

in the case when the operator A is positive-definite and self-adjoint. Starting from the positive definiteness of the operator A, S. G. Mikhlin, following K. Friedrichs, introduces an auxiliary Hilbert space H with scalar product

$$[u, v] = (Au, v), \qquad u,\ v \in D(A),$$

and norm

$$|u|^2 = [u, u] = (Au, u), \qquad u \in D(A).$$

S. G. Mikhlin makes wide use of this space in investigating convergence: both of the variational method and of other methods.

In the present note we show that the variational method can be generalized to a broader class of linear operators.

Let, in equation (1), the linear operator A be defined on a set dense in H and be representable in the form of a product

$$A = A_0 A_1, \tag{2}$$

where A_0 is a self-adjoint operator, A_1 a closed operator. We shall call an operator A having the representation (2) **positive in the generalized sense** if, for every nonzero element $u \in D(A)$, the inequality

$$(Au, A_1 u) > 0 \tag{3}$$

holds.

It follows from this definition that, if $(Au, A_1u) = 0$, then $u \equiv 0$.

For an operator positive in the generalized sense, by virtue of (2), the equality

$$(Au, A_1u) = (A_1u, Au), \qquad u \in D(A). \tag{4}$$

holds.

It is easy to show that, if A is an operator positive in the generalized sense, then equation (1) can have no more than one solution.

Let us construct the functional

$$\Phi(u) = (Au, A_1u) - (A_1u, f) - (f, A_1u), \qquad u \in D(A), \tag{5}$$

corresponding to equation (1). The generalization of the variational method proposed here is based on the following theorem.

Theorem 1. *If A is an operator positive in the generalized sense, then finding the solution of equation (1) is equivalent to finding an element $u \in D(A)$ that gives the minimum of the functional (5).*

If, in particular, $A_1 = E$, then the operator A will be positive in the usual sense, and from Theorem 1 there follows an analogous theorem for a positive operator, proved by S. G. Mikhlin in (^{1}).

Regarding the operator A as positive in the generalized sense, we cannot, generally speaking, assert the existence of a solution of equation (1). In order to ensure the existence of a solution of equation (1) and to find this solution by seeking the minimum of the functional (5), we impose on the operator A certain additional restrictions.

We shall call an operator A, having the representation (2), **positive-definite in the generalized sense** if, for every element $u \in D(A)$, the inequalities

$$(Au, A_1u) \geq \gamma^2 \|u\|^2, \qquad \gamma > 0, \tag{6}$$

$$\|A_1u\|^2 \leq C^2 (Au, A_1u), \qquad C > 0. \tag{7}$$

hold.

It is easy to show that a positive-definite operator in the generalized sense is at the same time an invertible* operator.

Let a positive-definite, in the generalized sense, operator A be given on a set $M = D(A)$ dense in H. Define on the set M the scalar product $[u, v]_1$ by the formula

$$[u, v]_1 = (Au, A_1 v), \qquad u, v \in M. \tag{8}$$

Then, as is easy to show, the usual axioms of a scalar product will be satisfied, and the linear set M may be regarded as a Hilbert (generally speaking, complex) space. We shall call the closure of the set M in the sense of the metric (8) the space H_1. It follows from this that M is dense in H_1. The norm of an element u in the space H_1 will be denoted by $|u|_1$, so that

$$|u|_1^2 = [u, u]_1 = (Au, A_1 u), \qquad u \in M. \tag{9}$$

Then inequalities (6) and (7) may be written in the form

$$\|u\| \leq \frac{1}{\gamma} |u|_1, \qquad u \in M, \tag{10}$$

$$\|A_1 u\| \leq C |u|_1, \qquad u \in M. \tag{11}$$

For the space H_1 the following two theorems hold.

Theorem 2. *All elements of the space H_1 also belong to the space H.*

In the proof of this theorem inequalities (10), (11) are used and the closedness of the operator A_1 is taken into account.

Using Theorem 2, inequalities (10) and (11) are not difficult to extend to the whole space H_1.

Theorem 3. *In order that the sequence $\{\varphi_n\}$, $\varphi_n \in D(A)$, be complete in H_1, it is sufficient that the sequence $\{A\varphi_n\}$ be complete in H.*

From Theorem 3 one easily obtains:

Corollary. *If H is separable, then H_1 is also separable.*

In particular, if $A_1 = E$, then inequalities (6) and (7) coincide, and a positive-definite in the generalized sense operator A becomes positive-definite in the usual sense. In this case the space H_1 coincides with the space H_0, and from Theorems 2 and 3 there follow analogous theorems for the space H_0, proved by S. G. Mikhlin in (1).**

It is now not difficult to give the solution of the above-mentioned minimum problem in the case when, in equation (1), the operator A is positive-definite

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* This is what we call an operator having a bounded inverse operator.

** Theorem 2 for the space H was first proved by K. Friedrichs.

in the generalized sense. By formula (9), $(A_1u, A_1u) = [u, u]_1$. Further, the scalar product (u, f), where $u \in D(A)$ and f is a fixed element of H, is, as is easy to see, a bounded functional in H_1. Then there exists a unique element $u_0 \in H_1$ such that

$$(u, f) = [u, u_0], \qquad u \in H_1. \tag{12}$$

Thus the functional (5) is extended to all of H_1 and takes the form

$$\Phi(u) = [u, u]_1 - [u, u_0]_1 - [u_0, u]_1.$$

Reducing this functional to the form $\Phi(u) = |u - u_0|_1^2 - |u_0|_1^2$, we conclude that the minimum of $\Phi(u)$ is attained for $u = u_0$, with $\min \Phi(u) = -|u_0|_1^2$.

Formula (12) defines a certain bounded operator $u_0 = Gf$, by means of which, for a given element $f \in H$, one can find the element u_0 that gives the minimum of the functional $\Phi(u)$ in H_1. It can also be shown that, in the case where the space H_1 is separable, the indicated element u_0 is expressed in the form of a series converging both in H_1 and in H:

$$u_0 = \sum_{n=1}^{\infty} (f, A_1\varphi_n)\varphi_n, \tag{13}$$

where $\{\varphi_n\}$ is any complete sequence orthonormal in H_1.

We shall call, as usual, a sequence $\{u_n\}$, $u_n \in D(A)$, minimizing for the functional (5) if $\lim_{u\to\infty} \Phi(u_n) = \inf \Phi(u)$. Just as in the case of a positive-definite self-adjoint operator, Theorem 4 is valid.

Theorem 4. If the operator A is positive-definite in the generalized sense, then every minimizing sequence for the functional (5) converges in the space H_1 to the element that gives the minimum of this functional.

To find an approximate solution of equation (1), starting from the problem of minimizing the functional (5), one can apply the well-known Ritz method. We shall call this method, in the case considered, the Ritz moment method. On the basis of formula (13), Theorem 5 is easily proved.

Theorem 5. If the operator A is positive-definite in the generalized sense, then the Ritz moment method for equation (1) converges in the space H_1.

It can also be shown that the approximate solutions of equation (1) found by the Ritz moment method form a minimizing sequence. This gives, by virtue of Theorem 4, another proof of convergence of the Ritz moment method. In the case of a positive-definite self-adjoint operator, such a proof is given in the book of S. G. Mikhlin (^{2}).

It is easy to see that the ordinary Ritz method and the method of least squares are special cases of the Ritz moment method, so that the theorems on convergence of the Ritz and least-squares methods proved by S. G. Mikhlin in ([1]) follow from Theorem 5.

As an application, let us establish convergence of the Ritz moment method for the equation

$$Au = \sum_{k=0}^{l-1} (-1)^k \frac{d^k}{dx^k} \left[p_{k+1}(x) \frac{d^{k+1}}{dx^{k+1}} \right] = f(x), \tag{14}$$

where $l = \dfrac{n+1}{2}$, n is odd, under the boundary conditions

$$u(a) = u'(a) = \dots = u^{(l-1)}(a) = 0, \tag{15}$$

$$u'(b) = u''(b) = \dots = u^{(l-1)}(b) = 0.$$

We shall assume that the coefficients $p_k(x)$ are nonnegative and are continuously differentiable the required number of times, with $p_l(x)$ strictly positive. Denote by M the set of functions $u(x) \in C^{(n)}(a, b)$ satisfying the boundary conditions (15). This set is, evidently, dense in $H = L_2(a, b)$. We shall show that on the set M the operator A is positive-definite in the generalized sense and invertible.

Put

$$A_1 u = \frac{du}{dx}, \qquad A_0 u = \sum_{k=0}^{l-1} (-1)^k \frac{d^k}{dx^k} \left[p_{k+1}(x) \frac{d^k u}{dx^k} \right].$$

The operator A_0 is, evidently, self-adjoint, and the operator A_1 is closed, since it is easy to see that $A_1^{**} = A_1$. Moreover $A = A_0 A_1$. Integrating by parts and taking account of the conditions (15), we obtain

$$(Au, A_1 u) = \sum_{k=0}^{l-1} \int_a^b p_{k+1}(x) \left(\frac{d^{k+1} u}{dx^{k+1}} \right)^2 dx \geq \int_a^b p_l(x) \left(\frac{d^l u}{dx^l} \right)^2 dx \geq p_0' \, \|u^{(l)}\|, \tag{16}$$

where $p_0' = \min p_l(x)$. Next, using the conditions (15), it is easy to obtain the inequalities

$$\|u^{(k)}\| \leq \frac{b-a}{\sqrt{2}} \, \|u^{(k+1)}\|, \qquad k = 0, 1, \dots, l-1. \tag{17}$$

Then, on the basis of the inequality following from (17),

$$\|u^{(l)}\| \geq \left(\frac{\sqrt{2}}{b-a}\right)^{l} \|u\|,$$

from (16) we obtain that

$$(Au, A_1 u) \geq \gamma^2 \|u\|^2, \qquad \gamma = \sqrt{p_0'} \left(\frac{\sqrt{2}}{b-a}\right)^{l}.$$

In addition, from (17) and (16) we also find

$$\|A_1 u\|^2 \leq C^2 (Au, A_1 u), \qquad C = \frac{1}{\sqrt{p_0'}} \left(\frac{b-a}{\sqrt{2}}\right)^{l-1}.$$

Thus, the operator A is positive-definite in the generalized sense.

By Theorem 5, the Ritz-moment method for equation (1) converges in the space H_1 corresponding to the operator A. It is easy to see that convergence of the Ritz-moment method for equation (1) in the space H_1 means convergence in the mean of derivatives of order $l = \dfrac{n+1}{2}$ of the approximate solutions. It may also be shown that the indicated convergence implies uniform convergence on the interval $a \leq x \leq b$ both of the approximate solutions u_n themselves and of their derivatives up to order $l-1$, inclusive.

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2. S. G. Mikhlin, *Direct Methods in Mathematical Physics*, 1950.

Note: Figure translations are in progress. See original paper for figures.