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Abstract

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ON A CLASS OF ITERATIVE PROCESSES FOR THE APPROXIMATE SOLUTION OF OPERATOR EQUATIONS

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For the approximate solution of the equation

$$P(x) = 0, \tag{1}$$

where P is a twice differentiable operator from a Banach space X into a normed space Y, several iterative processes are known which can be represented by the general formula

$$\Delta x_{n+1} = x_{n+1} - x_n = -(E + \alpha R_n)^{-1}[E + (\alpha + 1)R_n]\Gamma_{nP}(x_n), \tag{2}$$

where E is the identity operator, $\Gamma_n = [P'(x_n)]^{-1}$, $R_n = \frac{1}{2}\Gamma_n P''(x_n)\Gamma_{nP}(x_n)$, and α is a real number.

For example, for $\alpha = 0$ we obtain the "Chebyshev process" (see ([1])); for $\alpha = -1$, the "process of tangent hyperbolas" (see ([2])); for $\alpha = -2$, the process proposed in ([3]).

Concerning the convergence of the processes (2) to the solution of equation (1) in the case where the operator P is analytic, the following theorem holds.

Theorem 1. *If the following conditions are satisfied:*

1) *there exists the inverse operator* $\Gamma_0 = [P'(x_0)]^{-1}$, *and* $\|\Gamma_0 P(x)\| \leq \eta_0$;

2) *the operator* P *is analytic in the sphere*

$$\|x - x_0\| \leq \frac{\delta_0 \eta_0}{1 - s_0^2 h_0^2 (1 - h_0)}, \tag{3}$$

and

$$\left\| \frac{1}{j!} \Gamma_0 P^{(j)}(x_0) \right\| \leq A_0 H_0^{j-1} \quad (j = 2, 3, ...);$$

3) *the quantities* η_0, A_0, H_0 *satisfy the inequalities:*

$$|\alpha|A_0H_0\eta_0 < 1, \qquad |\alpha+1|A_0H_0\eta_0 < 1, \qquad h_0 = H_0\delta_0\eta_0 < 1,$$

where

$$\delta_0 = \frac{1-(|\alpha|-1)A_0H_0\eta_0}{1-|\alpha|A_0H_0\eta_0}, \qquad q_0 = 1 - A_0\frac{h_0(2-h_0)}{(1-h_0)^2} > 0,$$

$$p_0^2 = \frac{s_0^2h_0^2}{q_0(1-h_0)^2} \le 1,$$

$$s_0^2 = \frac{\left[|2+\alpha| - (|2\alpha+\alpha^2|-1)A_0H_0\eta_0\right](A_0/\delta_0)^2(1-h_0) + A_0\delta_0(1-|\alpha|A_0H_0\eta_0)^2}{q_0(1-h_0)^2(1-|\alpha|A_0H_0\eta_0)^2},$$

then equation (1) has in the sphere (3) a solution x^*, to which the process (2) converges with rate

$$\|x^* - x_n\| \le \frac{a_0^n p_0^{3^n-1}}{1-a_0p_0^6}\,\delta_0\eta_0 \quad (a_0 = q_0(1-h_0)^3). \tag{4}$$

Proof. We shall show that, when x_0 is replaced by x_1, conditions 1)–3) are not violated. Since Δx_1 can be written in the form

$$\Delta x_1 = -\Gamma_0 P(x_0) - (E+\alpha R_0)^{-1}R_0\Gamma_0 P(x_0),$$

we have

$$\|\Delta x_1\| \le \eta_0 + \frac{A_0H_0\eta_0^2}{1-|\alpha|A_0H_0\eta_0} = \delta_0\eta_0.$$

In view of the fact that x_1 belongs to the sphere (3), we obtain

$$\|\Gamma_0(P'(x_0)-P'(x_1))\| = \left\|\sum_{j=1}^{\infty}\frac{1}{j!}\Gamma_0P^{(1+j)}(x_0)\Delta x_1^j\right\| \le 1-q_0 < 1,$$

and, therefore, there exists

$$Q^{-1} = [E - \Gamma_0(P'(x_0)-P'(x_1))]^{-1},$$

with $\|Q^{-1}\| \le 1/q_0$. But then $\Gamma_1 = Q^{-1}\Gamma_0$ also exists and, consequently, condition 1) is satisfied when x_0 is replaced by x_1.

The estimates A_1, H_1, and η_1 are found, for example, as follows:

$$\left\|\frac{1}{j!}\Gamma_1 P^{(j)}(x_1)\right\| \le \frac{1}{q_0}\left\|\frac{1}{j!}\Gamma_0 P^{(j)}(x_0)\right\| + \frac{1}{q_0}\left\|\frac{1}{j!}\Gamma(P^{(j)}(x_1) - P^{(j)}(x_0))\right\| \le$$

$$\le \frac{1}{q_0}\left\{A_0 H_0^{j-1} + A_0 H_0^{j-1}\sum_{i=1}^{\infty}\frac{(j+i)!}{j!\,i!}h_0^i\right\} = \frac{A_0 H_0^{j-1}}{q_0(1-h_0)^{j+1}},$$

whence

$$A_1 = \frac{A_0}{q_0(1-h_0)^2}, \qquad H_1 = \frac{H_0}{1-h_0},$$

$$\Gamma_1 P(x_1) = \frac{1}{2}Q^{-1}\Gamma_0 P''(x_0)(E+\alpha R_0)^{-1}R_0\Gamma_0 P(x_0)\{(2+\alpha)E+$$

$$+(E+\alpha R_0)^{-1}R_0$$

$\Gamma 0P(x_0)$ + Q^{-1} {j=3}^{∞}$\frac{1}{j!}$Γ_0P^{(j)}$(x_0)\Delta\ x_1^j$,]

$$\|\Gamma_1 P(x_1)\| \le (1-h_0)s_0^2h_0^2\eta_0 = \eta_1.$$

The fact that condition 3) is satisfied for x_1 is now verified directly:

$$A_1H_1\eta_1 = p_0^2A_0H_0\eta_0 \le A_0H_0\eta_0,$$

$$\delta_1 = \frac{1-(|\alpha|-1)A_1H_1\eta_1}{1-|\alpha|A_1H_1\eta_1} \le \frac{1-(|\alpha|-1)A_0H_0\eta_0}{1-|\alpha|A_0H_0\eta_0} = \delta_0,$$

$$h_1 = H_1\delta_1\eta_1 = \frac{\delta_1}{\delta_0}s_0^2h_0^3 \le s_0^2h_0^3 \le h_0 < 1,$$

$$A_1\frac{h_1(2-h_1)}{(1-h_1)^2} = \frac{\delta_1}{\delta_0}p_0^2A_0h_0\frac{2-h_1}{(1-h_1)^2} \le A_0\frac{h_0(2-h_0)}{(1-h_0)^2}$$

and, thus, $q_1 \ge q_0 > 0$.

Finally, it is not difficult to prove that $p_1^2 \le p_0^6$ and, quite analogously,

$$(1-h_1)s_1^2h_1^2 \le p_0^4(1-h_0)s_0^2h_0^2. \tag{5}$$

The analyticity of the operator P in the sphere

$$\|x - x_1\| \leq \frac{\delta_1 \eta_1}{1 - s_1^2 h_1^2 (1 - h_1)} \tag{6}$$

follows from the fact that the sphere (6) is contained in (3). Indeed, if x belongs to (6), then

$$\|x - x_0\| \leq \|x - x_1\| + \|\Delta x_1\| \leq \frac{\delta_1 \eta_1}{1 - s_1^2 h_1^2 (1 - h_1)} + \delta_0 \eta_0 \leq$$

$$\leq \frac{\delta_0 \eta_1}{1 - s_0^2 h_0^2 (1 - h_0)} + \delta_0 \eta_0 = \frac{\delta_0 \eta_0}{1 - s_0^2 h_0^2 (1 - h_0)}.$$

Thus, all the conditions of the theorem are satisfied for x_1, and we can continue the definition of the elements x_n and the computation of the quantities associated with them, η_n, A_n, H_n, etc., by the formulas

$$\eta_{n+1} = (1 - h_n) s_n^2 h_n^2 \eta_n, \quad A_{n+1} = \frac{A_n}{q_n (1 - h_n)^2}, \quad H_{n+1} = \frac{H_n}{1 - h_n},$$

$$h_{n+1} \leqslant s_n^2 h_n^3, \quad \delta_{n-1} \leqslant \delta_n, \quad \rho_{n+1}^2 \leqslant \rho_n^6,$$

to which is added the inequality, also obtained from (5),

$$(1 - h_{n+1}) s_{n+1}^2 h_{n+1}^2 \leqslant \rho_n^4 (1 - h_n) s_n^2 h_n^2.$$

Repeated application of the last inequalities gives:

$$\rho_n^2 \leqslant \rho_0^{2 \cdot 3^n}, \quad (1 - h_n) s_n^2 h_n^2 \leqslant a_0 \rho_0^{2 \cdot 3^n}, \quad \eta_n \leqslant a_0^n \rho_0^{3^n - 1} \eta_0.$$

By virtue of this and of the obvious inequality $3^{n+p} - 1 \geqslant 3^n + 6(p-1)$ $(p, n = 1, 2, ...)$, we obtain

$$\|x_{n+p} - x_n\| \leqslant a_0^n \rho_0^{3^n - 1} \eta_0 \delta_0 \frac{1 - a_0^p \rho_0^{6p}}{1 - a_0 \rho_0^6}.$$

Thus the sequence $\{x_n\}$ converges, i.e., there exists an element $x^* = \lim_{n \to \infty} x_n$. Passing in the last inequality to the limit as $p \to \infty$, we obtain (4). Taking there $n = 0$ and taking into account the inequality $a_0 \rho_0^6 \leqslant (1 - h_0) s_0^2 h_0^2$, we see that all x_p, and also x^*, belong to (3).

Finally, we shall also prove that the element x^* obtained is a solution of equation (1). This is obtained by passing to the limit in the equality

$$P(x_n) + P'(x_n)[E + (\alpha + 1)R_n]^{-1}(E + \alpha R_n)(x_{n+1} - x_n) = 0.$$

Indeed, $\|x_{n+1} - x_n\| \to 0$, while $\|P'(x_n)[E + (\alpha + 1)R_n]^{-1}(E + \alpha R_n)\|$ is bounded. The theorem is proved.

When only the differentiability (up to and including the third order) of the operator P is known, the following theorem holds.

Theorem 2. *If the following conditions are satisfied:*

1) *there exists* Γ_0, *and*

$$\|\Gamma_0\| \leqslant B_0, \quad \|\Gamma_0 P(x_0)\| \leqslant \eta_0;$$

2) *in the sphere*

$$\|x - x_0\| \leqslant \frac{\varepsilon_0 \eta_0}{1 - g_0 r_0^2 k_0^2} \tag{7}$$

there exist $P''(x)$ *and* $P'''(x)$, *and*

$$\frac{1}{2}\|P''(x)\| \leqslant K, \quad \frac{1}{6}\|P'''(x)\| \leqslant L;$$

3) *the numbers* η_0, B_0, K *and* L *satisfy the inequalities*

$$|\alpha| k_0 = |\alpha| B_0 K \eta_0 < 1, \quad |\alpha + 1| k_0 < 1,$$

$$2k_0\varepsilon_0 = 2k_0 \frac{1 - (|\alpha| - 1)k_0}{1 - |\alpha| k_0} < 1, \quad l_0 \varepsilon_0 = B_0 L \eta_0^2 \varepsilon_0 \leqslant A k_0^2, \quad \nu_0 k_0 \leqslant 1,$$

where

$$\nu_0^2 = \frac{|2 + \alpha| + A - (|2\alpha + \alpha^2| - 1)k_0}{g_0^2 (1 - |\alpha| k_0)^2}, \quad g_0 = 1 - 2k_0 \varepsilon_0,$$

then equation (1) has in the sphere (7) a solution x^*, *to which process (2) converges with the rate*

$$\|x^* - x_n\| \leqslant \frac{g_0^n (\nu_0 k_0)^{3^n - 1} \varepsilon_0 \eta_0}{1 - g_0 (\nu_0 k_0)^6}.$$

The proof differs only slightly from that given in (4) for Newton’ s method.

Despite the great generality of Theorems 1 and 2, the conditions and convergence estimates for concrete processes obtained from them are no worse than the conditions and estimates in the previously known theorems for the same processes.

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Note: Figure translations are in progress. See original paper for figures.