GENERALIZED SOLUTIONS OF MONGE–AMPÈRE EQUATIONS

z-f(x_0,y_0)=p_0(x-x_0)+q_0(y-y_0)

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Abstract

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MATHEMATICS

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GENERALIZED SOLUTIONS OF MONGE–AMPÈRE EQUATIONS

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1. Let Φ be an arbitrary convex surface, given by the equation $z = f(x, y)$ in a domain D. To each point $(x_0, y_0) \in D$ we assign, in the plane of the variables (p, q), the set of points $\{(p_0, q_0)\}$ such that p_0 and q_0 are the coefficients in the equation of some supporting plane

$$z - f(x_0, y_0) = p_0(x - x_0) + q_0(y - y_0)$$

to Φ, passing through the point $(x_0, y_0, f(x_0, y_0))$. Thus there is defined a certain, generally speaking multivalued, mapping ψ of the domain D onto the plane (p, q), analogous in its properties to the spherical image of the surface Φ ([1]). Let $R(p, q)$ be a continuous function of the variables p, q on the whole plane, satisfying the condition $R(p, q) \geq R_0 = \text{const} > 0$. By the R-area of the normal image of a set $M \subset D$ we shall mean the number

$$\omega_R(M) = \iint_{\psi(M)} \frac{dp\, dq}{R(p, q)},$$

and the mapping ψ will be called the normal image of the surface Φ.

Theorem 1. *If, for a convex surface Φ, $\omega_R(D) < +\infty$, then $\omega_R(M)$ is a nonnegative completely additive set function on the ring of Borel sets of the domain D.*

If $R(p, q) = (1 + p^2 + q^2)^{3/2}$, then the R-area of the normal image becomes simply the area of the spherical image of the surface Φ.

2. Consider, in some convex domain D in the (x, y)-plane, the Monge–Ampère equation

$$rt - s^2 = \varphi(x, y)R(p, q), \tag{1}$$

where $\varphi(x, y) > 0$ and $R(p, q) \geq R_0 = \text{const} > 0$ are continuous functions, the first in the closed domain D, and the second in the plane p, q. Let $z(x, y)$ be a twice continuously differentiable solution in D of equation (1). Then the graph of the function $z(x,y)$ will be a convex surface Φ, which has at each point a unique tangent (or, what is the same, supporting plane), and moreover at all points of Φ the tangent planes are distinct. Therefore the normal image of the surface Φ, defined by the formulas $p = \partial z/\partial x,\ q = \partial z/\partial y$, will be one-to-one.

Let $M \subseteq D$ be an arbitrary Borel set; then

$$\iint_M \varphi(x,y)\,dx\,dy = \iint_{\psi(M)} \frac{dp\,dq}{R(p,q)} = \omega_R(M). \tag{2}$$

This equality shows that a solution of equation (1) determines a convex surface Φ with a prescribed R-area of the normal image on the ring of Borel sets. Thus, with equation (1) one can associate a certain R-area of the normal image, and the problem of integrating equation (1) can be treated as the problem of finding a convex surface with a prescribed R-area of the normal image.

We shall call a function $z(x,y)$, determining a convex surface Φ, a **generalized solution of equation (1)** if, for every inner subdomain M of the domain D, equality (2) is satisfied.

The problem of finding a convex surface with a prescribed R-area of the normal image is usefully considered while imposing on the surface a certain boundary condition*. In the present work two types of such conditions will be considered.

3. Let us first consider infinite convex surfaces. The behavior of such a surface at infinity is characterized by its limit cone ([2]).

Theorem 2. *Let there be given on the plane (x,y) a completely additive non-negative set function $\mu(M)$ and a convex cone K, projecting one-to-one onto the whole plane (x,y). Then, if*

$$\mu(P_{x,y}) = \iint_{\psi(K)} \frac{dp\,dq}{R(p,q)},$$

where $P_{x,y}$ is the plane (x,y), $\psi(K)$ is the normal image of the cone K, and the function $R(p,q)$ satisfies the conditions of § 1, then there exists an infinite convex surface having K as its limit cone, and having the set function $\mu(M)$ as its R-area of the normal image.

Theorem 2 is proved by a passage to the limit from polyhedra. For polyhedra, a completely additive set function $\mu(M)$ is a function of point loads, and the limit cone K is a polyhedral angle. Theorem 2 for polyhedra is proved with the aid of A. D. Aleksandrov’ s “mapping lemma” ([2]).

Theorem 3. *If, under the conditions of Theorem 2, $\mu(P_{x,y}) = 0$, then the surface with the given limit cone K and the prescribed R-area of the normal image is determined up to a parallel displacement in a direction perpendicular to the plane (x,y). If, however, $\mu(P_{x,y}) = 0$, then K is a dihedral angle and the surface may be any cylinder whose limit cone is the given dihedral angle.*

4. The Dirichlet problem in the generalized formulation is stated as follows: Let, in a convex domain D with boundary Γ, there be given a completely additive nonnegative set function $\mu(M)$. Does there then exist a convex surface Φ, passing through some curve L with projection Γ onto $P_{x,y}$, projecting one-to-one onto the open domain D, for which, in every inner subdomain M of the domain D, the equality $\omega_R(M) = \mu(D)$ holds? (It is admitted that the surface which is a solution of the generalized Dirichlet problem may project non-one-to-one onto the curve Γ.)

Theorem 4. *If*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dp\,dq}{R(p,q)} = +\infty,$$

then the Dirichlet problem in the generalized formulation is always solvable.

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* In the case $R(p,q) = (1 + p^2 + q^2)^{3/2}$, these problems were solved by A. D. Aleksandrov (^{2}) by a purely geometric method.

Theorem 5. If

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dp\,dq}{R(p,q)} = \Omega(R) < +\infty,$$

then, for the solvability of the Dirichlet problem in the generalized formulation, it is necessary that the nonnegative completely additive set function $\mu(M)$ satisfy the condition $\mu(D) \leq \Omega(R)$, and it is sufficient that $\mu(D) < \Omega(R)$. (Here D is understood as the interior of the convex domain in which the Dirichlet problem is posed.)

Theorem 6. Consider convex surfaces Φ, turned with their convexity toward $z < 0$, and which are generalized solutions of one and the same Dirichlet problem in a convex domain D. Denote by $z(x, y)$ the function defining the surface Φ inside D, and put

$$z_\Phi(P) = \lim_{(x,y)\to P} z(x, y),$$

where (x, y) is an interior point and P is a boundary point of the domain D. Under the conditions of Theorems 4 and 5 there exists one and only one generalized solution of the Dirichlet problem for which, at any point $P \in \Gamma$, the quantity $z_\Phi(P)$ is not smaller than the analogous quantity for any other generalized solution of the same problem.

The proofs of Theorems 4 and 5 are carried out by passage to the limit from polyhedra, by the same method as Theorem 2.

All the results connected with generalized solutions of equations (1) carry over to the case of functions of n variables ($n \geq 2$). In this case the equation has the form

$$H(z) = \varphi(x_1, x_2, \dots, x_n) \cdot R(p_1, \dots, p_n),$$

where $H(z)$ is the Hessian of the function $z(x_1, \dots, x_n)$; $p_i = \partial z / \partial x_i$; $\varphi(x_1, \dots, x_n) > 0$ and $R(p_1, \dots, p_n) \geq \text{const} > 0$ are continuous functions, the first of the variables $x_1, \dots, x_n$ in some convex n-dimensional domain D, and the second in the plane $p_1, p_2, \dots, p_n$.

5. If it is assumed that the functions $\varphi(x, y) \geq \text{const} > 0$ and $R(p, q) \geq \text{const} > 0$ are three times continuously differentiable, the first in the convex domain D, and the second in any circle in the p, q-plane, then the following theorem holds.

Theorem 7. Let the function $z(x, y)$ be a generalized solution of the Dirichlet problem for equation (1) in the convex domain D. Then, if with respect to the functions $\varphi(x, y)$ and $R(p, q)$ the conditions formulated in Sec. 5 are fulfilled, the function $z(x, y)$ is three times continuously differentiable inside D and at all points of the open domain D satisfies equation (1).

From Theorem 7 follows the regularity of a convex surface whose spherical-image area on the plane x, y is given as a function of sets by the formula

$$\omega(M) = \iint_M \varphi(x, y)\, dx\, dy,$$

where $\varphi(x, y) \geq \varphi_0 = \text{const} > 0$, $R(p, q) \geq R_0 = \text{const} > 0$ are three times continuously differentiable. This result was formulated by A. D. Aleksandrov in ([2]) without proof, which until now has not been known.

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REFERENCES

1. A. D. Aleksandrov, *Intrinsic Geometry of Convex Surfaces*, 1948.

2. A. D. Aleksandrov, *Convex Polyhedra*, 1950.

Note: Figure translations are in progress. See original paper for figures.