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Abstract

Full Text

MATHEMATICS

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A BOUNDARY-VALUE PROBLEM FOR THE EQUATION $\psi_{\sigma\sigma} - K(\sigma)\psi_{\theta\theta} = 0$ WITH DATA ON A CHARACTERISTIC AND ON THE STRAIGHT LINES $\sigma = \text{const}$

(Presented by Academician V. I. Smirnov on 5 XI 1956)

1. Consider the equation

$$L(\psi) \equiv \psi_{\sigma\sigma} - K(\sigma)\psi_{\theta\theta} = 0, \quad K(\sigma) \in C^{(2)} \quad (1_0)$$

inside a hyperbolic strip ($K(\sigma) > 0$). Let AB be a segment of a characteristic of one family with endpoints at the points (σ_0, θ_0) and (σ_1, θ_1) . We shall find and investigate the solution of the boundary-value problem

$$L(\psi) = 0; \quad \psi|_{AB} = \bar{\psi}(\sigma); \quad \psi|_{\sigma=\sigma_0} = \bar{\psi}(\sigma_0); \quad \psi|_{\sigma=\sigma_1} = \bar{\psi}(\sigma_1) \quad (I_0)$$

in the strip $S\{\sigma_0 \leq \sigma \leq \sigma_1; -\infty < \theta < +\infty\}$.

For the time being suppose that on $[\sigma_0, \sigma_1]$ $\bar{\psi}(\sigma) \in C$ and has bounded variation. Below the conditions on $\bar{\psi}(\sigma)$ and $K(\sigma)$ will be specified more precisely.

Problem (I_0) contains, in particular, the gas-dynamical problem with data on a characteristic and a free surface.

2. We make the following transformations:

a) After subtracting from the solution of equation (1_0) the solution

$$\psi_0 = \frac{1}{\sigma_1 - \sigma_0} \{ \sigma_1 \bar{\psi}(\sigma_0) - \sigma_0 \bar{\psi}(\sigma_1) + \sigma \bar{\psi}(\sigma_1) - \sigma \bar{\psi}(\sigma_0) \},$$

the problem for $\psi^* = \psi - \psi_0$ takes the form

$$L(\psi^*) = 0; \quad \psi^*|_{AB} = \bar{\psi}(\sigma) - \psi_0|_{AB} = \bar{\psi}^*(\sigma); \quad \psi^*|_{\sigma=\sigma_0} = \psi^*|_{\sigma=\sigma_1} = 0. \quad (I_1)$$

b) The substitution $dt = \sqrt{K} d\sigma$, $t(\sigma_0) = 0$, carries (1₀) into the equation

$$\psi_{tt} - \psi_{\theta\theta} + \frac{K_1'}{2K_1} \psi_t = 0, \quad K_1 = K_1(t) = K(\sigma). \quad (1)$$

c) Introducing, instead of ψ^* , $u = \psi^* K_1^{1/4}$, we obtain from (1₁)

$$u_{tt} - u_{\theta\theta} + q(t)u = 0, \quad q(t) = -K_1^{-1/4} \frac{d^2 K_1^{1/4}}{dt^2}. \quad (1_2)$$

d) Finally, for convenience we normalize the problem, replacing (t, θ) by (ζ, ϑ) according to the formulas $t_1 \zeta = t$; $t_1 \vartheta = +(\theta - \theta_0)$; $t_1 = t(\sigma_1)$. In this case the point B passes into $(1, 1)$. We now have

$$u(t, \theta) = v(\zeta, \vartheta); \quad M(v) \equiv v_{\zeta\zeta} - v_{\vartheta\vartheta} + N(\zeta)v = 0, \quad (1)$$

where $N(\zeta) = t_1^2 q(\zeta t_1)$, $N(\zeta) \in C[0, 1]$.

Problem (I₁) is reduced to the form

$$M(v) = 0; \quad v|_{\vartheta=\zeta} = \psi^*(\sigma) K_1^{1/4}(\sigma) = p(\zeta); \quad v|_{\zeta=0} = v|_{\zeta=1} = 0, \quad (I)$$

where $p(\zeta)$ is continuous on $[0, 1]$, has bounded variation, and vanishes at the endpoints.

3. To solve problem (I) we shall use the family of particular solutions of equation (1), which is obtained in the usual way by separation of the variables ζ, ϑ . Consider the series $T = \sum_n c_n B_n(\zeta) \exp(i\mu_n \vartheta)$, in which c_n are as yet undetermined complex numbers; μ_n and $B_n(\zeta)$ are the eigenvalues and normalized eigenfunctions of the Sturm-Liouville problem:

$$B_n'' + [\mu_n^2 + N(\zeta)]B_n = 0, \quad (2)$$

$$B_n(0) = B_n(1) = 0. \quad (3)$$

If, in general, the values $p(\zeta)$ of the desired function are prescribed on some curve $\vartheta = f(\zeta)$, then an attempt formally to satisfy this boundary condition by a function of the form T leads to an expansion of $p(\zeta)$ in a series in the eigenfunctions of the following non-self-adjoint problem:

$$z_n'' - 2i\mu_n f' z_n' + z_n \{N - i\mu_n f'' + \mu_n^2(1 - f'^2)\} = 0, \quad (4)$$

$$z_n(0) = z_n(1) = 0. \quad (5)$$

Equation (4) is obtained if in (2) one substitutes the function $B_n(\zeta) = z_n(\zeta) \exp[-i\mu_n f(\zeta)]$. The study of problem (4), (5) for various $f(\zeta)$ is of independent interest. In our case, however, $f(\zeta) \equiv \zeta$, and equation (4) is considerably simplified:

$$z_n'' - 2i\mu_n z_n' + Nz_n = 0. \quad (6)$$

The following theorem will now play an essential role.

Theorem 1. Let μ_n and $z_n(\zeta) = B_n(\zeta) \exp(i\mu_n \zeta)$ be the eigenvalues and eigenfunctions of problem (6), (5), where $N(\zeta) \in C$, $B_n(\zeta)$ are normalized, and $\mu_0 = 0$ is not an eigenvalue. If $p(\zeta)$ has bounded variation on $[0, 1]$, then

$$\sum_{n=-\infty}^{+\infty} \frac{iz_n(\zeta)}{\mu_n} \int_0^1 p(\tau) \overline{z_n'(\tau)} d\tau = \frac{1}{2} \{p(\zeta + 0) + p(\zeta - 0) - p(0+) - p(1-)\}. \quad (7)$$

This assertion follows directly from a theorem proved by Misho ⁽¹⁾.

4. Thus, in view of Theorem 1, we shall seek the solution of problem (I) in the form

$$v = \sum_{n=-\infty}^{+\infty} c_n B_n(\zeta) \exp(i\mu_n \vartheta), \quad (8)$$

where

$$\mu_{-n} = -\mu_n, \quad B_{-n}(\zeta) = -B_n(\zeta); \quad (9)$$

μ_n and $B_n(\zeta)$ are determined from (2), (3).

For $\mu_0 = 0$ the general solution of equation (2) has the form

$$B_0 = K_1^{1/4} \left(c_1 + c_2 \int_0^t \frac{dt}{\sqrt{K_1}} \right),$$

and it is easy to see that $\mu_0 = 0$ is not an eigenvalue.

Assuming the uniform convergence of the series (8), to be proved below, in the strip S under consideration, we see that the boundary conditions on the straight lines

$\zeta = 0$, $\zeta = 1$ are satisfied by virtue of (3), while on the characteristic AB we obtain the expansion of $p(\zeta)$ in the series

$$p(\zeta) = \sum_{n=-\infty}^{+\infty} c_n z_n(\zeta), \quad (10)$$

where $z_n(\zeta) = y_n(\zeta) + ix_n(\zeta) = B_n(\zeta) \exp(i\mu_n\zeta)$ are the eigenfunctions of problem (6), (5). For the $p(\zeta)$ under consideration, expansion (10) is justified by Theorem 1. The coefficients c_n , as is seen from (7), are determined by the formula

$$c_n = \frac{i}{\mu_n} \int_0^1 p(y'_n - ix'_n) d\zeta.$$

Denoting

$$a_n = 2 \int_0^1 p(\zeta) x'_n(\zeta) d\zeta, \quad b_n = 2 \int_0^1 p(\zeta) y'_n(\zeta) d\zeta,$$

we finally obtain

$$v = \sum_{n=1}^{\infty} \frac{B_n(\zeta)}{\mu_n} (a_n \cos \mu_n \vartheta - b_n \sin \mu_n \vartheta). \quad (11)$$

Here the imaginary part of the sum (8) vanishes by virtue of (9).

5. Let us investigate the series (11) under the following additional conditions on $p(\zeta)$ and $N(\zeta)$ in $[0, 1]$: $p(\zeta) \in C^{(2)}$, and $p'''(\zeta)$ and $N'(\zeta)$ exist and have bounded variation. We shall use the asymptotic formulas for μ_n and $B_n(\zeta)$, which are easily obtained by the usual method of successive approximations (see, for example, (2)):

$$\mu_n = \pi n - \frac{H(0, 1)}{\pi n} + \frac{O(1)}{n^3}; \quad (12)$$

$$B_n(\zeta) = \sqrt{2} \left\{ \sin \pi n \zeta + h(\zeta) \frac{\cos \pi n \zeta}{\pi n} - \left[h^2(\zeta) + \frac{N(\zeta)}{2} - H(0, 1) \right] \frac{\sin \pi n \zeta}{2\pi^2 n^2} \right\} + \frac{O(1)}{n^3}, \quad (13)$$

where

$$H(x, y) = \frac{1}{2} \int_x^y N(\zeta) d\zeta; \quad h(\zeta) = H(0, \zeta) - \zeta H(0, 1).$$

In the class of $p(\zeta)$ under consideration we obtain

$$a_n = \frac{a}{n^2} + \frac{O(1)}{n^3}, \quad b_n = \frac{b}{n} + \frac{O(1)}{n^3}, \quad (14)$$

$$b = \frac{1}{\pi\sqrt{2}} \left\{ p'(1) - p'(0) + \int_0^1 p(\zeta)N(\zeta) d\zeta \right\}; \quad (15)$$

$$a = \frac{1}{2\pi^2\sqrt{2}} \left\{ p''(1) - p''(0) - 2p'(1)H(0, 1) - \int_0^1 p(\zeta) [N'(\zeta) + 2N(\zeta)H(0, \zeta)] d\zeta \right\}. \quad (16)$$

Now (11) can be written in the form

$$v = \frac{b}{\pi\sqrt{2}} \left\{ \sum_{n=1}^{\infty} \frac{\cos \pi n(\zeta + \vartheta)}{n^2} - \sum_{n=1}^{\infty} \frac{\cos \pi n(\zeta - \vartheta)}{n^2} \right\} + \sum_{n=1}^{\infty} \frac{\alpha(n, \zeta, \vartheta)}{n^3}, \quad (17)$$

where

$$\alpha(n, \zeta, \vartheta) = \frac{n^3 B_n(\zeta)}{\mu_n} (a_n \cos \mu_n \vartheta - b_n \sin \mu_n \vartheta) + \frac{nb\sqrt{2}}{\pi} \sin \pi n \zeta \sin \pi n \vartheta.$$

Draw from the points A and B characteristics of the second family ($\zeta + \vartheta = \text{const}$) until they meet the opposite walls, respectively at the points A_1 and B_1 ; then from the points A_1 and B_1 draw characteristics of the first family A_1A_2 and B_1B_2 , and so on. We obtain a broken line L , consisting of segments of characteristics, which divides S into triangles S_k . In each of them the sums of the first two series in (17) have an elementary form. Denote these sums, respectively, by $Q_1^{(k)}$ and $Q_2^{(k)}$. For example, in $\triangle ABA_1$, $Q_1^{(0)} - Q_2^{(0)} = \pi^2 \vartheta (\zeta - 1)$. In passing through L , the normal derivative of the difference $Q_1^{(k)} - Q_2^{(k)}$ has a discontinuity. With the aid of the asymptotic formulas (12)–(14) it is not difficult to verify that

$$\frac{\partial^p \alpha(n, \zeta, \vartheta)}{\partial \zeta^i \partial \vartheta^{p-i}} = O(n^p) \quad (p = 0, 1, 2; i = 0, \dots, p),$$

where

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n^3} \frac{\partial^2 \alpha}{\partial \zeta^i \partial \vartheta^{2-i}} = \\ & = \gamma_1(\zeta, \vartheta) \sum_{n=1}^{\infty} \frac{\sin \pi n(\zeta + \vartheta)}{n} + \gamma_2(\zeta, \vartheta) \sum_{n=1}^{\infty} \frac{\sin \pi n(\zeta - \vartheta)}{n} + \sum_{n=1}^{\infty} \frac{O(1)}{n^2}, \end{aligned}$$

$\gamma_1(\zeta, \vartheta)$ and $\gamma_2(\zeta, \vartheta)$ are continuous in S and contain a and b homogeneously linearly.

We formulate the results.

Theorem 2. If on $[0, 1]$ $N(\zeta) \in C$, $p(\zeta) \in C^{(2)}$, there exist $N'(\zeta)$ and $p''(\zeta)$, both of bounded variation, and $p(0) = p(1) = 0$, then the function

$$v = \frac{b}{\pi\sqrt{2}} \left\{ Q_1^{(k)}(\zeta, \vartheta) - Q_2^{(k)}(\zeta, \vartheta) + \sum_{n=1}^{\infty} \frac{\alpha(n, \zeta, \vartheta)}{n^3} \right\} \quad (18)$$

gives a solution of problem (I), twice continuously differentiable at any point $M \in S$ not lying on L .

Theorem 3. For the existence of a classical solution of problem (I) in the whole strip S , the condition $a = b = 0$ is necessary. If, together with it, the conditions of Theorem 2 are satisfied, this solution is given by formula (11).

Theorem 4. If on $[0, 1]$ $N(\zeta) \in C$, $p(\zeta) \in C$, there exists $p'(\zeta)$ of bounded variation, and $p(0) = p(1) = 0$, then the series (11) converges uniformly in S and gives a “generalized solution of problem (I) from the class L_2 ” in the sense of the definition given in the work ⁽⁴⁾.

The uniqueness of such a solution has been proved.

Another way of obtaining a generalized solution of the problem under consideration in the strip S , with the aid of the Fourier transform in the space of generalized functions, is indicated in the work ⁽³⁾.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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