ON THE PERIODICITY OF A JET EMERGING FROM A SYMMETRIC NOZZLE UNDER DESIGN CONDITIONS

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

HYDROMECHANICS

G. A. DOMBROVSKII

ON THE PERIODICITY OF A JET EMERGING FROM A SYMMETRIC NOZZLE UNDER DESIGN CONDITIONS

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1. In the present article the plane problem of the supersonic outflow of gas from a symmetric nozzle under design conditions is considered, and the question of the periodicity of the jet is investigated in the case where there are no shock waves in the flow (Fig. 1). The investigation is carried out by approximate analytical methods, based on replacing, in the equations

$$\frac{\partial\varphi}{\partial\xi} = -\sqrt{K_1}\frac{\partial\psi}{\partial\xi}, \qquad \frac{\partial\varphi}{\partial\eta} = \sqrt{K_1}\frac{\partial\psi}{\partial\eta} \tag{1}$$

the Chaplygin function $K_1(t)$ for adiabatic flow (^{1}) by functions of a simpler form. In equations (1), φ is the velocity potential, ψ is the stream function,

Fig. 1

$$\xi = \frac{1}{2}(t-\vartheta), \qquad \eta = \frac{1}{2}(t+\vartheta) \tag{2}$$

are characteristic variables, where ϑ is the angle of inclination of the velocity vector to the X-axis and t is the variable replacing the magnitude of the velocity.

We assume the motion of the gas in the nozzle to be known; moreover, in the region $A_0B_0C_0$, whose boundary consists of two characteristics of different families issuing from B_0, and of the segment A_0C_0 of the X-axis, the Jacobian

$$J = \frac{\partial(\varphi,\psi)}{\partial(v,\vartheta)} = \left[K_1^{-1/2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - K_1^{1/2}\left(\frac{\partial\psi}{\partial t}\right)^2\right]\frac{dt}{dv} = -K_1^{1/2}\frac{\partial\psi}{\partial\xi}\frac{\partial\psi}{\partial\eta}\frac{dt}{dv} \tag{3}$$

takes finite values and is different from zero.

Owing to symmetry with respect to the X-axis, it is sufficient to consider only the upper half of the flow; to the gas flow under consideration to the right of the characteristic A_0B_0 there corresponds, in the (t, ϑ)-plane, the domain shown in Fig. 2. It is bounded by segments of characteristics

$$\begin{aligned} \xi = \xi_1 &= \frac{1}{2}(t^* + \vartheta^*), \qquad & \xi = \xi_2 &= \frac{1}{2}(t^* - \vartheta^*), \\ \eta = \eta_1 &= \frac{1}{2}(t^* + \vartheta^*), \qquad & \eta = \eta_2 &= \frac{1}{2}(t^* - \vartheta^*), \end{aligned} \tag{4}$$

where t^* is the constant value of t on the surface of the jet and ϑ^* is the angle of inclination of the velocity vector at the exit from the nozzle at the point B_0. The corresponding points in Figs. 1 and 2 are denoted identically.

2. For $K_1 = \text{const}$ we have

$$\psi = f_1(\xi) - f_2(\eta), \tag{5}$$

where $f_1(\xi)$ and $f_2(\eta)$ are arbitrary functions of the characteristic variables. These functions are determined by successive solution

boundary-value problems with known data on the characteristic and a condition on the free surface, and problems with known data on the characteristic and the condition $\psi = 0$ for $\vartheta = 0$ (^{2}). As a result we obtain the following solutions, respectively, for regions 0—4:

$$f_1^{(0)}(\xi) = f(\xi), \qquad f_2^{(0)}(\eta) = f(\eta); \tag{6}$$

$$f_1^{(1)}(\xi) = f(\xi), \qquad f_2^{(1)}(\eta) = f(t^* - \eta) + f(\xi_1) - f(\xi_2); \tag{7}$$

$$f_1^{(2)}(\xi) = f(t^* - \xi), \qquad f_2^{(2)}(\eta) = f(t^* - \eta); \tag{8}$$

$$f_1^{(3)}(\xi) = f(t^* - \xi), \qquad f_2^{(3)}(\eta) = f(\eta) + f(\xi_1) - f(\xi_2); \tag{9}$$

$$f_1^{(4)}(\xi) = f(\xi), \qquad f_2^{(4)}(\eta) = f(\eta), \tag{10}$$

where f is a known function.

Fig. 2

Comparing (6) and (10), we are convinced that for $K_1 = \text{const}$, i.e., when the adiabat in the pressure–specific volume plane is replaced by a straight line, the gas jet issuing from the nozzle has a strictly periodic character.

The accuracy of the solution of the problem will increase if one applies the approximate method of S. A. Khristianovich (^{3}), in which it is assumed that

$$K_1 = \text{const} \cdot t^4, \tag{11}$$

and therefore for the stream function we have

$$\psi = \frac{f_1(\xi) - f_2(\eta)}{\xi + \eta}. \tag{12}$$

It is easy to verify that the functions (6)—(10) are solutions of the problem under consideration also in this case, and, consequently, the flow in the jet repeats periodically.

For sufficiently small values of ϑ^*, when the two approximations of the adiabat considered above can be successfully applied, the jet in reality has a structure close to periodic.

3. Let us now show that the application of the method proposed earlier by the author ($^{1-7}$) does not lead to a periodic solution and, moreover, that a continuous potential flow cannot exist at all at a sufficiently large distance from the nozzle.

In the proposed approximate method

$$K_1 = (n \operatorname{tg} mt)^4 \tag{13}$$

and, correspondingly,

$$\psi = \frac{1}{n}\left\{m[-f_1(\xi) + f_2(\eta)] + \frac{\operatorname{ctg} mt}{2}\,[-f_1'(\xi) + f_2'(\eta)]\right\}, \tag{14}$$

where n and m are arbitrary constants, whose values are established from the condition of the best approximation to the adiabatic flow of the gas. The approximation of the adiabat is obtained more accurately than in the methods considered above, and, in particular, one can obtain third-order contact of the adiabat and the pressure—density curve corresponding to a fictitious gas.

The sought functions $f_1(\xi)$ and $f_2(\eta)$ are determined by successive solution of boundary-value problems for regions 1—4 (^{5,7}). In the region $A_0B_0C_0$ (region 0) we have

$$f_1^{(0)}(\xi) = f(\xi), \qquad f_2^{(0)}(\eta) = f(\eta), \tag{15}$$

where

$$f(\eta) = 2n\cos^2 m(\xi_2+\eta)\int_{\eta_2}^{\eta}\frac{\sin m(\xi_2+\eta)}{\cos^3 m(\xi_2+\eta)}\,\widetilde{\psi}(\eta)\,d\eta, \tag{16}$$

where on the characteristic A_0B_0 the stream function assumes the known values $\psi = \widetilde{\psi}(\eta)$.

In the region $B_0C_0D_0$

$$f_1^{(1)}(\xi) = f(\xi), \qquad f_2^{(1)}(\eta) = -f(t^*-\eta)+2be^{-b\eta}\int_{\eta_1}^{\eta} f(t^*-\eta)e^{b\eta}\,d\eta + c_1e^{-b\eta}+c_2, \tag{17}$$

where

$$c_1 = \frac{f'(\xi_2)-f'(\xi_1)+2bf(\xi_2)}{be^{-b\eta_1}}, \tag{18}$$

$$c_2 = [f(\xi_1)-f(\xi_2)] + \frac{1}{b}[f'(\xi_1)-f'(\xi_2)]. \tag{19}$$

Here and below, to shorten the notation, the designation

$$b = 2m\,\mathrm{tg}\,mt^*$$

is adopted.

In the region $C_0D_0A_1$

$$f_1^{(2)}(\xi) = f_2^{(1)}(\xi), \qquad f_2^{(2)}(\eta) = f_2^{(1)}(\eta). \tag{20}$$

In the region $D_0A_1B_1$

$$f_1^{(3)}(\xi) = f_1^{(2)}(\xi), \qquad f_2^{(3)}(\eta) = f(\eta) + c_3e^{-b\eta} + 2c_2, \tag{21}$$

where

$$c_3 = \int_{\eta_1}^{\eta_2}[F(\eta)+F(t^*-\eta)]e^{b\eta}\,d\eta, \qquad F(\eta) = f'(\eta)+\frac{1}{b}f''(\eta). \tag{22}$$

Finally, in the region $A_1B_1C_1$

$$f_1^{(4)}(\xi) = f(\xi) + c_3e^{-b\xi}, \qquad f_2^{(4)}(\eta) = f(\eta) + c_3e^{-b\eta}. \tag{23}$$

Taking the functions (23) as the initial ones, one may, analogously to the preceding case, obtain the solutions for region 8 in the form

$$f_1^{(8)}(\xi) = f(\xi) + 2c_3 e^{-b\xi}, \qquad f_2^{(8)}(\eta) = f(\eta) + 2c_3 e^{-b\eta}, \tag{24}$$

and, consequently, in the general case for the region $4k$ $(k = 1, 2, \ldots)$ we have

$$f_1^{(4k)}(\xi) = f(\xi) + kc_3 e^{-b\xi}, \qquad f_2^{(4k)}(\eta) = f(\eta) + kc_3 e^{-b\eta}. \tag{25}$$

If the derivative $d\widetilde{\psi}/d\eta$, like the Jacobian J, preserves its sign on A_0B_0, then with the aid of (16) it is easy to establish that the function $F(\eta)$ on A_0B_0 also preserves its sign and is not identically equal to zero. Therefore $c_3 \neq 0$, and, in contrast to the previous results, a nonperiodic solution is obtained for the jet.

Examining the Jacobian

$$J = \frac{\partial(\varphi, \psi)}{\partial(v, \vartheta)} = \left[\frac{\operatorname{ctg}^2 mt}{n^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - n^2 \operatorname{tg}^2 mt \left(\frac{\partial\psi}{\partial t}\right)^2\right]\frac{dt}{dv}, \tag{26}$$

one can show that for large values of k the solutions obtained lose their physical meaning. Indeed, at the point $B_k(\xi = \xi_2, \eta = \eta_1)$

$$J_{B_k} = \left\{\left[-m\cos 2mt^* f'(\eta_1) + \frac{\sin 2mt^*}{2} f''(\eta_1)\right] + kbmc_3\left(e^{-b\eta_1} - e^{-b\xi_2}\right)\right\} \times$$

$$\times \left\{\left[mf'(\eta_1) + \frac{\sin 2mt^*}{2} f''(\xi_2)\right] - kbmc_3\left(e^{-b\eta_1} - e^{-b\xi_2}\right)\right\}\frac{dt/dv}{\sin^2 2mt^*}, \tag{27}$$

and, for sufficiently large k, this expression becomes negative, whereas on A_kC_k the Jacobian is greater than zero (here $\partial\psi/\partial t = 0$ and always $dt/dv > 0$). Thus, on some line the Jacobian necessarily vanishes.

Mathematical Institute
named after V. A. Steklov
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.