MATHEMATICS

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Abstract

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MATHEMATICS

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ON NONLINEAR OSCILLATIONS WITH ONE DEGREE OF FREEDOM OF A SYSTEM WITH SLOWLY VARYING PARAMETERS

(Presented by Academician N. N. Bogolyubov, 4 VI 1957)

§ 1. **Statement of the problem.** In ([3],[4]) there were investigated the oscillatory solutions of the equation

$$\ddot{x} + Q(\varepsilon t, x) + \varepsilon f(\varepsilon, \varepsilon t, x, \dot{x}) = 0, \qquad |\varepsilon| \ll 1, \tag{1}$$

where Q and εf were interpreted respectively as a slowly varying (because of the presence of the small factor ε in the argument t) principal force, causing the oscillatory motion, and a small perturbing force depending on the velocity $\dot{x}$. In paper ([3]) equation (1) was written in the form

$$\frac{d}{dt}\left[m(\varepsilon t)\dot{x}\right] + \varepsilon f + Q = 0$$

($m(\varepsilon t)$ is a slowly varying mass), which is equivalent to (1). It was shown that, under the condition $\operatorname{sign} Q = \operatorname{sign} x$, which determines the oscillatory character of the solution, and under some other restrictions, the solution of (1) satisfying the initial conditions $x(0) = x_0,\ \dot{x}(0) = \dot{x}_0\ (x_0^2 + \dot{x}_0^2 \neq 0)$, on a large interval of time $t \sim 1/\varepsilon$, oscillates about the equilibrium position $x = 0$ with slowly varying amplitude and period, for which formulas were obtained determining these quantities with accuracy up to quantities of order ε inclusive on the interval $t \sim 1/\varepsilon$.

The amplitude of the oscillations is described by two amplitude curves

$$F_1(\varepsilon t, \varepsilon) \equiv F_{10}(\varepsilon t) + \varepsilon F_{11}(\varepsilon t), \qquad F_2(\varepsilon t, \varepsilon) \equiv F_{20}(\varepsilon t) + \varepsilon F_{21}(\varepsilon t),$$

to which, as $\varepsilon \to 0$, the maxima and minima of the solution respectively approach. The functions $F_{10}(\varepsilon t)$ and $F_{20}(\varepsilon t)$ are the zero approximations of the amplitude, describing the maxima and minima of the solution with an error $\sim \varepsilon$ on the interval $t \sim 1/\varepsilon$. The quantities $\varepsilon F_{11}(\varepsilon t)$ and $\varepsilon F_{21}(\varepsilon t)$ are corrections to the zero approximation of the amplitude, while $F_1(\varepsilon, \varepsilon t)$ and $F_2(\varepsilon, \varepsilon t)$ are first approximations describing the amplitude with accuracy up to ε inclusive. For $F_{kj}(\varepsilon t)$ $(k = 1, 2;\ j = 0, 1)$, in ([3],[4]) equations were derived which determine these quantities as slow functions of time t, and which we do not write out again here. The period of oscillations was defined in [3,4] as the time between two neighboring maxima or minima. The period is described by means of the function

$$T(\varepsilon t,\varepsilon)\equiv T_0(\varepsilon t,F_{10},F_{20})+\varepsilon T_1(\varepsilon t,F_{10},F_{20},F_{11},F_{21}),$$

where

$$T_0\equiv 2\sum_{k=1}^{2}(-1)^{k+1}\int_0^{F_{k0}}dx\left(2\int_x^{F_{k0}}Q(\varepsilon t,y)\,dy\right)^{-1/2}, \tag{2}$$

$$T_1\equiv\sum_{k=1}^{2}\left\{\frac{\partial T_0}{\partial F_{k0}}F_{k1}+(-1)^k\sum_{i=1}^{2}\int_0^{F_{k0}}dx\left(2\int_x^{F_{k0}}Q(\varepsilon t,y)\,dy\right)^{-3/2}\times\right.$$

$$\left.\times\int_x^{F_{k0}}f\left[0,\varepsilon t,x,(-1)^i\left(2\int_x^{F_{k0}}Q(\varepsilon t,y)\,dy\right)^{1/2}\right]dx\right\}.$$

and $F_{kj}=F_{kj}(\varepsilon t)$ $(k=1,2;\ j=0,1)$ are approximations to the amplitude. For an arbitrary value of t within the period under consideration, the quantity T_0 is the zeroth approximation to the period, determining it with an error $\sim\varepsilon$, and εT_1 is the correction to the zeroth approximation, since $T=T_0+\varepsilon T_1$ is the first approximation to the period and determines it with accuracy up to and including ε, if in the term T_0 one substitutes $t=\theta$, where θ is the midpoint of the period under consideration. The method [3,4] also makes it possible to compute the period and amplitude with any higher accuracy.

Knowledge of the amplitude and period of the solution (1) does not yet give a direct possibility of computing the value of the solution $x(t,\varepsilon)$ for an arbitrary value of t from the time interval of order $1/\varepsilon$. This problem is solved in the present paper. Here a formula is derived for the solution of (1) which determines it with an error $\sim\varepsilon$. The methods developed make it possible to compute the solution also with greater, prescribed in advance, accuracy. The problem of studying equations containing a small parameter was suggested to the author by A. N. Tikhonov.

§ 2. **Asymptotic formula for the solution.** The solution $x(t,\varepsilon)$ of equation (1), satisfying the initial conditions $x(0)=x_0$, $\dot{x}(0)=\dot{x}_0$, and its derivative $\dot{x}(t,\varepsilon)$ for an arbitrary value of t from the interval $t\sim 1/\varepsilon$, are determined with an error $\sim\varepsilon$ by the formulas

$$\int_{x(t,\varepsilon)}^{F_{20}(\varepsilon t)}dx\left(2\int_x^{F_{20}(\varepsilon t)}Q(\varepsilon t,y)\,dy\right)^{-1/2}=T_0(\varepsilon t)\left|\varphi(\varepsilon,\varepsilon t)-E[\varphi(\varepsilon,\varepsilon t)]-\frac{1}{2}\right|, \tag{3}$$

$$\dot{x}(t,\varepsilon) = \left(2\int_{x(t,\varepsilon)}^{F_{20}(\varepsilon t)} Q(\varepsilon t, x)\,dx\right)^{-1/2} \operatorname{sign}\left\{\varphi(\varepsilon,\varepsilon t) - E[\varphi(\varepsilon,\varepsilon t)] - \frac{1}{2}\right\}, \quad (4)$$

where

$$\varphi(\varepsilon,\varepsilon t) = \int_0^t \frac{dt}{T_0(\varepsilon t)+\varepsilon T_1(\varepsilon t)} - \frac{1}{T_0(0)}\int_{x_0}^{F_{10}(0)} dx\left(2\int_x^{F_{10}(0)} Q(0,y)\,dy\right)^{-1/2} \alpha(\dot{x}_0), \quad (5)$$

$$\alpha = \operatorname{sign}\dot{x}_0 \quad \text{when } \dot{x}_0 \neq 0, \qquad \alpha = 1 \quad \text{when } \dot{x}_0 = 0.$$

In formulas (3)–(5) the quantities F_{kj}, T_j ($k = 1,2;\ j = 0,1$) are approximations to the amplitude and the period (see § 1); x_0 and $\dot{x}_0$ are the initial values; the quantity $x(t,\varepsilon)$ entering into (4) must be computed by formula (3).

The convenience of formulas (3)–(5) and of the formulas for $F_{kj}(\varepsilon t)$ ($k = 1,2;\ j = 0,1$) from ([3,4]) consists in the fact that all the quantities being computed are expressed directly through the functions Q, f entering into (1).

The conditions under which the results presented are valid are analogous to those restrictions which are given in ([3,4]), and we do not repeat them here. We note, however, that the restrictions in ([3,4]) may be substantially weakened; thus, for example, one can in general dispense with the requirement $\partial Q/\partial x\big|_{x=0} \neq 0$, given in ([3]).

§ 3. **On the study of equation (1) by means of the averaging method.** The solutions of (1) can also be investigated by means of the averaging method of N. M. Krylov–N. N. Bogolyubov ([2]). Such an investigation was carried out by Yu. A. Mitropolsky ([1]). For comparison of the results of the present paper and ([1]), let us write formulas (3)–(5) from § 2 in another form. If the general solution of the degenerate equation is known explicitly as a function of t,

$$\ddot{x}_0 + Q(\tau, x_0) = 0, \quad (6)$$

in which the argument τ, standing in place of εt, is regarded as a parameter, then formulas (3)–(5) can be written with the aid of this solution. Consider the solution (6) satisfying the initial conditions $x_0(0) = b_1 > 0$, $\dot{x}_0(0) = 0$. Under the restrictions imposed on Q, this solution, for the corresponding initial values, will be periodic and can be

is found by quadratures. Obviously, b_1 is the maximum of the solution. Denote by $b_2 < 0$ the minimum x_0. The quantities b_1, b_2 are connected by the relation

$\int_{b_1}^{b_2} Q\,dx = 0$, and the solution $x_0 = x_0(b_1, b_2, \tau, t)$ has period $T_0(\tau, b_1, b_2)$, where T_0 is the function defined by formula (2), in which, instead of $\varepsilon t, F_{10}, F_{20}$, respectively τ, b_1, b_2 are substituted. Then formulas (3)–(5) may be written in the equivalent form

$$\begin{aligned} x(t) &= x_0[F_{10}(\varepsilon t),\, F_{20}(\varepsilon t),\, \varepsilon t,\, \psi(t,\varepsilon)], \\ \dot{x}(t) &= \dot{x}_0[F_{10}(\varepsilon t),\, F_{20}(\varepsilon t),\, \varepsilon t,\, \psi(t,\varepsilon)], \end{aligned} \tag{7}$$

where $\psi(t,\varepsilon) = T_0(\varepsilon t)\varphi(\varepsilon, \varepsilon t)$. Thus, formulas (3)–(5), by which the solution of (1) is computed, express it, according to (7), with an error $\sim \varepsilon$, in terms of the solution $x_0(b_1, b_2, \tau, t)$ of the degenerate equation (6), into which, in place of b_1, b_2, τ and the argument t, are substituted respectively $F_{10}(\varepsilon t)$, $F_{20}(\varepsilon t)$, εt, and the function ψ, which is the phase of the oscillations.

A representation of the solution in a form analogous to (7) was applied to (1), where equation (1) was written in the form

$$\frac{d^2x}{dt^2} + f(x,\tau) = \varepsilon F\left(\tau, x, \frac{dx}{dt}, \varepsilon\right) \quad (\tau = \varepsilon t). \tag{8}$$

To obtain the asymptotics of the solution of (8), in [1] the degenerate equation $d^2x/dt^2 + f(x,\tau) = 0$ is introduced (τ is a parameter). The solution of this degenerate equation is written in the form $x = z(a, \omega t + \varphi, \tau)$, where a and φ are arbitrary constants; $\omega(a,\tau)$ is the angular frequency; z is a function periodic with period 2π in the argument $\psi = \omega t + \varphi$, satisfying the equation $\omega^2 z''_{\psi^2} + f(z,\tau) = 0$. In [1] the approximate equations are not written directly in terms of the functions f, F entering (8), and, in order to obtain the asymptotics, it is necessary to know the general solution of the equation $\omega^2 z''_{\psi^2} + f(z,\tau) = 0$ in the form of an expansion in a trigonometric Fourier series of the form $z(a,\psi,\tau) = a\sin\psi + \sum_{n\neq 1}\theta_n(a,\tau)e^{in\psi}$.

Thus, as the parameter a determining the solution, the amplitude of the first harmonic is chosen (let us note that in nonlinear equations the amplitude of the first harmonic, in contrast to equations close to linear ones, differs, generally speaking, from the true amplitude, describing the maximum deviations from the equilibrium position, by a finite amount). For the solution of (8), in [1] the formula is given

$$x = z(a_1, \psi_1, \tau) \quad (\tau = \varepsilon t), \tag{9}$$

where a_1 is the amplitude of the first harmonic and ψ_1 is the phase of the oscillations, determined by a certain system of differential equations, the right-hand sides of which are expressed through an infinite sequence of Fourier coefficients of the function $z(a,\psi,\tau)$ and of certain other functions. The mentioned system of equations determines a_1 with an error $\sim \varepsilon$, and the phase ψ_1 with a certain finite, generally speaking, error on the interval $t \sim 1/\varepsilon$. Therefore formula (9) does not, generally speaking, determine the solution of (8) with an error $\sim \varepsilon$ on the interval $t \sim 1/\varepsilon$, but gives a certain finite error. What has been said is confirmed by the example of an equation of the form $\ddot{x} + Q(x) = \varepsilon^2 \dot{x}$, where the function Q is such that $\omega'_a \neq 0$. The calculation of this example is especially simplified if one sets specifically $Q = \operatorname{sign} x$. For equations close to linear ones, i.e. for $Q = \omega^2(\varepsilon t)x$, formula (9) gives an error $\sim \varepsilon$, since in this case the frequency ω does not depend on a; in the nonlinear case, however, ω, generally speaking, depends essentially on a. The accuracy

the accuracy of formula (9) for the nonlinear case can be increased if, in the equations for a_1 and ψ_1, terms $\sim \varepsilon^2$ are retained.

In (1) the case is also considered in which the function F from (8) depends explicitly periodically on t. In the equations determining a_1 and ψ_1, given for this case in (1), terms $\sim \varepsilon^2$ have likewise not been retained.

Formulas (3)–(5) of § 2 are derived by means of the methods developed in ([3,4]), independently of the averaging method. If, as the parameter determining the solution, one chooses not the amplitude of the first harmonic but the true amplitude of the oscillations, then the successive application of the averaging method in combination with certain methods from ([3,4]) also leads to formulas (3)–(5).

The author takes this opportunity to express his gratitude to L. S. Solov' ev for interesting discussions.

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CITED LITERATURE

[1] Yu. A. Mitropol' skii, *Nonstationary Processes in Nonlinear Oscillatory Systems*, 1955.
[2] N. N. Bogolyubov, Yu. A. Mitropol' skii, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, 1956.
[3] V. M. Volosov, DAN, **106**, No. 1 (1956).
[4] V. M. Volosov, DAN, **114**, No. 6* (1957); **115**, No. 1 (1957).

* CORRECTION

In my article "On Solutions of Nonlinear Differential Equations of the Second Order with Slowly Varying Coefficients," printed in DAN, vol. 114, No. 6, the following corrections should be made.

P. 1151, line 10 from the bottom, printed:

$$\times \int_x^{F_{j0}} dy \left(2\int_y^{F_{j0}} Q(\varepsilon t, z)\, dz\right)^{-3/2} \int_y^{F_{j0}} f_1 \left[\varepsilon t, z, (-1)^i \left(2\int_z^{F_{j0}} Q(\varepsilon t, u)\, du\right)^{1/2}\right] dz -$$

should read:

$$\times \left(2\int_x^{F_{j0}} Q(\varepsilon t, z)\, dz\right)^{-3/2} \int_x^{F_{j0}} f_1 \left[\varepsilon t, z, (-1)^i \left(2\int_z^{F_{j0}} Q(\varepsilon t, u)\, du\right)^{1/2}\right] dz -$$

P. 1151, line 9 from the bottom, printed:

$$-(-1)^{i+j} f'_{1x} \left[\varepsilon t, x, (-1)^i \left(2\int_x^{F_{j0}} Q(\varepsilon t, y)\, dy\right)^{1/2}\right] \left(2\int_x^{F_{j0}} Q(\varepsilon t, y)\, dy\right)^{-3/2} \times$$

should read:

$$-(-1)^{i+j} f'_{1x} \left[\varepsilon t, x, (-1)^i \left(2\int_x^{F_{j0}} Q(\varepsilon t, y)\, dy\right)^{1/2}\right] \left(2\int_x^{F_{j0}} Q(\varepsilon t, y)\, dy\right)^{-1/2} \times$$

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Note: Figure translations are in progress. See original paper for figures.