# MATHEMATICS

G. I. NATANSON

1957

SovietRxiv

### Abstract

### Full Text

MATHEMATICS

G. I. NATANSON

# ON A THEOREM OF S. M. LOZINSKII

*(Presented by Academician V. I. Smirnov on 23 V 1957)*

1. S. M. Lozinskii proved ([1]) the following theorem.

**Theorem.** Let the matrix of multipliers $\{\rho_m^{(n)}\}$ $(n = 0, 1, 2, \dots;\ m = 0, 1, 2, \dots, n)$ be such that for every $f(x) \in C_{2\pi}$, uniformly on $(-\infty, +\infty)$, one has

$$\lim_{n\to\infty} \left[\rho_0^{(n)} a_0 + \sum_{m=1}^{n} \rho_m^{(n)} (a_m \cos mx + b_m \sin mx)\right] = f(x)$$

($a_m, b_m$ are the Fourier coefficients of $f(x)$). Then, if $f(x) \in C_{2\pi}$ and

$$T(x) = a_0^{(n)} + \sum_{m=1}^{n} \left(a_m^{(n)} \cos mx + b_m^{(n)} \sin mx\right)$$

is a polynomial coinciding with $f(x)$ at the nodes $x_k = \dfrac{2k\pi}{2n+1}$ $(k = 0, 1, 2, \dots, 2n)$, then

$$\lim_{n\to\infty} \left[\rho_0^{(n)} a_0^{(n)} + \sum_{m=1}^{n} \rho_m^{(n)} \left(a_m^{(n)} \cos mx + b_m^{(n)} \sin mx\right)\right] = f(x)$$

uniformly on $(-\infty, +\infty)$.

2. This theorem carries over to the theory of series in ultraspherical polynomials. Namely, the following theorem is valid:

**Theorem.** Let $-\frac{1}{2} \le \alpha \le \frac{1}{2}$ and let $J_n(x) \equiv J_n^{(\alpha)}(x)$ be polynomials orthonormal on $[-1, 1]$ with weight $p(x) = (1 - x^2)^\alpha$. Let the matrix $\{\rho_m^{(n)}\}$ $(n = 0, 1, 2, \dots;\ m = 0, 1, 2, \dots, n)$ be such that for every $f(x) \in C([-1, 1])$ one has

$$S_n^{(\rho)}[f; x] = \sum_{m=0}^{n} \rho_m^{(n)} a_m J_m(x) \longrightarrow f(x) \quad n \to \infty$$

uniformly on $[-1+h,\, 1-h]$, where

$$a_m = \int_{-1}^{1} f(t) J_m(t) p(t)\, dt, \qquad h \in [0,1).$$

For the function $f(x) \in C([-1,1])$, form the interpolation polynomial $L_n[f;x]$, coinciding with $f(x)$ at the roots $x_k^{(n)}$ of the polynomial $J_n(x)$. Expand $L_n[f;x]$ in the polynomials $J_m(x)$:

$$L_n[f;x] = \sum_{m=0}^{n-1} a_m^{(n)} J_m(x)$$

and form

$$L_n^{(\rho)}[f;x] = \sum_{m=0}^{n-1} \rho_m^{(n)} a_m^{(n)} J_m(x).$$

Then

$$\lim_{n\to\infty} L_n^{(\rho)}[f;x] = f(x)$$

uniformly on $[-1+h,\, 1-h]$.

**3. Lemma.** If $T(x)$ is an even trigonometric polynomial of degree not exceeding $n$ and $\delta > 0$, then

$$\int_0^{\pi} |T(x)|\, dx \le 6(\pi\gamma)^{\delta} n^{\delta} \int_0^{\pi} \sin^{\delta} x\, |T(x)|\, dx,$$

where $\gamma = [\ln(4/3)]^{-1}$.

**Proof.** Obviously,

$$T(x) = T\left(x + \frac{1}{2\gamma n}\right) - \sum_{k=1}^{\infty} \frac{T^{(k)}(x)}{(2\gamma n)^k k!},$$

$$T(x) = T\left(x - \frac{1}{2\gamma n}\right) - \sum_{k=1}^{\infty} \frac{(-1)^k T^{(k)}(x)}{(2\gamma n)^k k!}.$$

Consequently,

$$\int_0^{\pi/2} |T(x)|\,dx \le \int_0^{\pi/2} \left|T\left(x+\frac{1}{2\gamma n}\right)\right|\,dx + \sum_{k=1}^{\infty} \frac{1}{(2\gamma n)^k k!} \int_0^{\pi} |T^{(k)}(x)|\,dx, \quad (1)$$

$$\int_{\pi/2}^{\pi} |T(x)|\,dx \le \int_{\pi/2}^{\pi} \left|T\left(x-\frac{1}{2\gamma n}\right)\right|\,dx + \sum_{k=1}^{\infty} \frac{1}{(2\gamma n)^k k!} \int_0^{\pi} |T^{(k)}(x)|\,dx. \quad (2)$$

Applying $k$ times Zygmund' s inequality (see, for example, ([2]), p. 566), we find

$$\int_0^{\pi} |T^{(k)}(x)|\,dx \le (2n)^k \int_0^{\pi} |T(x)|\,dx.$$

Hence

$$\sum_{k=1}^{\infty} \frac{1}{(2\gamma n)^k k!} \int_0^{\pi} |T^{(k)}(x)|\,dx \le (e^{1/\gamma}-1) \int_0^{\pi} |T(x)|\,dx = \frac{1}{3} \int_0^{\pi} |T(x)|\,dx. \quad (3)$$

Adding (1) and (2) and using (3), we obtain

$$\int_0^{\pi} |T(x)|\,dx - \frac{2}{3} \int_0^{\pi} |T(x)|\,dx \le \int_{1/2\gamma n}^{\pi/2+1/2\gamma n} |T(x)|\,dx + \int_{\pi/2-1/2\gamma n}^{\pi-1/2\gamma n} |T(x)|\,dx,$$

or

$$\int_0^{\pi} |T(x)|\,dx \le 6 \int_{1/2\gamma n}^{\pi-1/2\gamma n} |T(x)|\,dx.$$

For $x \in \left[\frac{1}{2\gamma n},\, \pi - \frac{1}{2\gamma n}\right]$ we have $\sin x \ge \frac{1}{\pi\gamma n}$. Therefore

$$\int_0^{\pi} |T(x)|\,dx \le 6 \int_{1/2\gamma n}^{1/2\gamma n} (\sin x \cdot \pi\gamma n)^{\delta} |T(x)|\,dx,$$

whence the lemma follows.

**4.** Passing to the proof of the theorem, let us note that $S_n^{(\rho)}[f;x]$ is a linear operator from the space $C([-1,1])$ into the space $C([-1+h,\, 1-h])$ with norm

$$\|S_n^{(\rho)}\| = \max_{x\in[-1+h,\,1-h]} \int_{-1}^{1} \left| \sum_{m=0}^{n} \rho_m^{(n)} J_m(x) J_m(t) \right| \rho(t)\, dt. \tag{4}$$

By the well-known Banach theorem,

$$\left\| S_n^{(\rho)} \right\| = O(1). \tag{5}$$

Put

$$l_k^{(n)}(x) = \frac{J_n(x)}{(x - x_k^{(n)}) J_n'(x_k^{(n)})}.$$

Then $(J_n(x_k^{(n)}) = 0)$

$$l_k^{(n)}(x) = -\left[ J_{n+1}(x_k^{(n)}) J_n'(x_k^{(n)}) \sqrt{\lambda_{n+1}} \right]^{-1} \sqrt{\lambda_{n+1}}\, \frac{J_n(x) J_{n+1}(x_k^{(n)}) - J_{n+1}(x) J_n(x_k^{(n)})}{x_k^{(n)} - x},$$

where

$$\lambda_{n+1} = \frac{(n+1)(2\alpha + n + 1)}{(2\alpha + 2n + 1)(2\alpha + 2n + 3)} = \frac{1}{4} + O(n^{-1}).$$

Denoting

$$\left[ J_{n+1}(x_k^{(n)}) J_n'(x_k^{(n)}) \sqrt{\lambda_{n+1}} \right]^{-1} = q_k^{(n)},$$

with the aid of the Christoffel-Darboux formula we obtain

$$l_k^{(n)}(x) = -q_k^{(n)} \sum_{m=0}^{n} J_m(x_k^{(n)}) J_m(x).$$

Consequently,

$$L_n^{(\rho)}[f; x] = -\sum_{k=1}^{n} q_k^{(n)} f(x_k^{(n)}) \sum_{m=0}^{n} \rho_m^{(n)} J_m(x_k^{(n)}) J_m(x),$$

$$\left\| L_n^{(\rho)} \right\| = \max_{x\in[-1+h,\,1-h]} \sum_{k=1}^{n} \left| q_k^{(n)} \sum_{m=0}^{n} \rho_m^{(n)} J_m(x_k^{(n)}) J_m(x) \right|. \tag{6}$$

For ultraspherical polynomials the following formulas hold (see, for example, (3), pp. 83 and 232):

$$(n+1)\sqrt{\frac{(n+2\alpha+1)(2n+2\alpha+1)}{(n+1)(2n+2\alpha+3)}}\,J_{n+1}(x) =$$

$$= (n+2\alpha+1)xJ_n(x) - (1-x^2)J_n'(x),$$

$$\left|J_n'(x_k^{(n)})\right|^{-1} = O\left(\frac{k^{\alpha+3/2}}{n^{\alpha+5/2}}\right) \qquad \left(0 \le x_{[n/2]}^{(n)} < \cdots < x_2^{(n)} < x_1^{(n)} < 1\right).$$

Hence, putting $x_k^{(n)} = \cos\theta_k$, we find

$$q_k^{(n)} = \frac{O(1)\,k^{2\alpha+3}}{n^{2\alpha+4}\sin^2\theta_k}.$$

For $\theta_k$ the Markov-Stieltjes inequalities are known (see (4), p. 40):

$$\frac{2k-1}{2n+1}\pi \le \theta_k \le \frac{2k}{2n+1}\pi.$$

Thus, for $1 \le k \le [n/2]$,

$$q_k^{(n)} = O(n^{-1})\sin^{2\alpha+1}\theta_k. \tag{7}$$

By symmetry of $J_n(x)$, (7) is also valid for $[n/2] \le k \le n$. Let the sum on the right-hand side of (6) attain its maximum at $x = x_0 \in [-1+h,\,1-h]$. Introducing the notation

$$\sum_{m=0}^{n} \rho_m^{(n)} J_m(\cos\theta) J_m(x_0) = T(\theta)$$

($T(\theta)$ is an even trigonometric polynomial of order $n$), we have

$$\left\|L_n^{(\rho)}\right\| = O(n^{-1})\sum_{k=1}^{n}\sin^{2\alpha+1}\theta_k\,|T(\theta_k)| = O(1)\sum_{k=-n}^{n}\frac{\left|\sin^{2\alpha+1}\theta_k\,T(\theta_k)\right|}{2n+1},$$

where $\theta_{-k} = -\theta_k$.

If $s_n(\theta)$ is a trigonometric polynomial of best approximation to $\sin^{2\alpha+1}\theta$ of order not exceeding $n$, then

$$\sin^{2\alpha+1}\theta = s_n(\theta) + O\left(n^{-2\alpha-1}\right), \tag{8}$$

$$\left\|L_n^{(\rho)}\right\| = O(1) \sum_{k=-n}^{n} \frac{|s_n(\theta_k)T(\theta_k)|}{2n+1} + O\left(n^{-2\alpha-1}\right) \sum_{k=-n}^{n} \frac{|T(\theta_k)|}{2n+1}.$$

Exactly as in (2), p. 568, it is proved that

$$\sum_{k=-n}^{n} \frac{|s_n(\theta_k)T(\theta_k)|}{2n+1} = O(1) \int_{-\pi}^{\pi} |s_n(\theta)T(\theta)|\ d\theta,$$

$$\sum_{k=-n}^{n} \frac{|T(\theta_k)|}{2n+1} = O(1) \int_{-\pi}^{\pi} |T(\theta)|\ d\theta.$$

Therefore

$$\left\|L_n^{(\rho)}\right\| = O(1) \int_{-\pi}^{\pi} |s_n(\theta)T(\theta)|\ d\theta + O\left(n^{-2\alpha-1}\right) \int_{-\pi}^{\pi} |T(\theta)|\ d\theta,$$

and, by virtue of (8) and the evenness of $T(\theta)$,

$$\left\|L_n^{(\rho)}\right\| = O(1) \int_{0}^{\pi} \sin^{2\alpha+1}\theta\ |T(\theta)|\ d\theta + O\left(n^{-2\alpha-1}\right) \int_{0}^{\pi} |T(\theta)|\ d\theta.$$

Applying the lemma, we find

$$\left\|L_n^{(\rho)}\right\| = O(1) \int_{0}^{\pi} |T(\theta)| \sin^{2\alpha+1}\theta\, d\theta = O(1) \int_{-1}^{1} \left| \sum_{m=0}^{n} \rho_m^{(n)} J_m(t) J_m(x_0) \right| \rho(t)\, dt.$$

Thus, by virtue of (4) and (5), the norms of the linear operators $L_n^{(\rho)}[f;x]$ are uniformly bounded. It remains to observe that, if $P(x)$ is an algebraic polynomial, then $\lim_{n\to\infty} L_n^{(\rho)}[P;x] = P(x)$, since for $n$ greater than the degree of $P(x)$,

$$L_n^{(\rho)}[P;x] = S_n^{(\rho)}[P;x].$$

Starting from estimates for the deviation of $S_n^{(\rho)}[f;x]$ from $f(x)$, one can obtain convergent estimates also for the deviation of $L_n^{(\rho)}[f;x]$ from $f(x)$. We formulate the simplest result of this kind.

Let, for every $f(x) \in C([-1,1])$,

$$\left|f(x) - S_n^{(\rho)}[f;x]\right| < A\omega_f(\varphi(n)) \qquad (-1+h \le x \le 1-h),$$

where $\omega_f$ is the modulus of continuity of $f(x)$ on $[-1,1]$; the quantity $\dfrac{1}{n\varphi(n)}$ is bounded; $\lim\limits_{n\to\infty} \varphi(n) = 0$; and $A$ and $\varphi$ do not depend on $n$, $x$, and $f(x)$. Then

$$\left|f(x) - L_n^{(\rho)}[f;x]\right| < B\omega_f(\varphi(n)) \qquad (-1+h \le x \le 1-h),$$

where $B$ likewise does not depend on $n$, $x$, or $f(x)$.

There is also a trigonometric analogue of this fact.

Leningrad State Pedagogical Institute
named after A. I. Herzen

Received
21 V 1957

## CITED LITERATURE

1. S. M. Lozinskii, *Mat. sbornik*, **14** (56), 175 (1944).

2. I. P. Natanson, *Constructive Theory of Functions*, Moscow-Leningrad, 1949.

3. G. Szegő, *Orthogonal Polynomials*, *Am. Math. Soc. Coll. Publ.*, **23**, 1939.

4. Ya. L. Geronimus, *Theory of Orthogonal Polynomials*, Moscow-Leningrad, 1950.

*Note: Figure translations are in progress. See original paper for figures.*