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Abstract

Full Text

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NONHOMOGENEOUS STRICTLY MARKOV PROCESSES

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For the case of processes homogeneous in time, the concept of a strictly Markov process was introduced by A. A. Yushkevich and the author in ([1]). There it was also shown that the known continuity conditions on the trajectories and transition probabilities of a Markov process imply the strict Markov property. Some special cases of these results were obtained independently of us by Hunt ([2]) and Reem ([3]). In the present note the general concept of a strictly Markov process is analyzed without the assumption that the process is homogeneous in time.

1. A Markov process is defined by specifying: 1) an interval I of the number line; 2) a set E (the phase space), and a certain σ-algebra $\mathfrak{B}$ of subsets of E; 3) a set Ω (the set of elementary events); 4) a function $x(t,\omega)$ $(t \in I,\ \omega \in \Omega)$ with values in E; 5) a system of probability measures $P_{s,x}$ $(s \in I,\ x \in E)$: the measure $P_{s,x}$ is defined on the σ-algebra $\mathfrak{M}^s$ generated by the ω-sets $\{x(t,\omega) \in \Gamma\}$ $(t \in I,\ t \geq s,\ \Gamma \in \mathfrak{B})$, and satisfies the condition $P_{s,x}\{x(s,\omega) = x\} = 1$.

These elements define a Markov process if:

(J_0) The function

$$P(s,x;\,t,\Gamma) = P_{s,x}\{x_t \in \Gamma\} \qquad (s < t \in I,\ \Gamma \in \mathfrak{B})$$

is $\mathfrak{B}$-measurable with respect to x.

(S_0) Whatever $s < t < v$ from I, $x \in E$, and $\Gamma \in \mathfrak{B}$ may be, for almost all $\omega \in \Omega$,

$$P_{s,x}\{x_v \in \Gamma \mid x_u,\ s \leq u \leq t\} = P(t, x_t;\, v, \Gamma)$$

($P_{s,x}\{- \mid x_u,\ s \leq u \leq t\}$ denotes conditional probability with respect to the σ-algebra $\mathfrak{M}_{s,t}$ generated by the ω-sets $\{x(u,\omega) \in \Gamma\}$, where $u \in [s,t]$, $\Gamma \in \mathfrak{B}$).

2. A function $\tau(\omega)$ on some subset Ω_τ of the space Ω, taking values in the interval I, will be called a random variable independent of the future and of the s-past if: 1) $\tau(\omega) \geq s$ for all $\omega \in \Omega_\tau$; 2) $\{\tau(\omega) \leq t\} \in \mathfrak{M}_{s,t}$ for all $t \geq s$.

We denote by $\mathfrak{M}_{s,\tau}$ the totality of all $A \subseteq \Omega_\tau$ such that $A \cap \{\tau \le t\} \in \mathfrak{M}_{s,t}$ for every $t \ge s$. The system $\mathfrak{M}_{s,t}$ is a σ-algebra in the space Ω_τ. Let $A \in \mathfrak{M}^s$ and $A \subseteq \Omega_\tau$. We shall write $P_{s,x}\{- \mid x_u,\ s \le u \le \tau\}$ instead of $P_{s,x}\{- \mid \mathfrak{M}_{s,\tau}\}$.

We give two definitions of strictly Markov processes. In the first definition we shall consider only variables $\tau(\omega)$ defined on all of Ω. The notation $\tau \le t$ means that $\tau(\omega) \le t$ for all $\omega \in \Omega$.

Definition 1. A Markov process is called strictly Markov in the first sense if it is measurable and satisfies the conditions:

(J_1) The function $P(s, x;\, t, \Gamma)$ is jointly measurable in s and x (with respect to the σ-algebra $\mathfrak{B}_I \times \mathfrak{B}$, where $\mathfrak{B}_I$ is the σ-algebra generated by the open subsets of the interval I).

(S_1) Whatever $s < t \in I$, $x \in E$, $\Gamma \in \mathfrak{B}$, and the random variable $\tau \le t$, independent of the future and the s-past, may be, for almost all $\omega \in \Omega$ (in the sense of the measure $P_{s,x}$) the equality

$$P_{s,x}\{x_t \in \Gamma \mid x_u,\ s \le u \le \tau\} = P(\tau, x; t, \Gamma)$$

holds.

Definition 2. A Markov process is called strongly Markov in the second sense if it is measurable and satisfies the following conditions:

(J_2) The function $P(s, x; t, \Gamma)$ is jointly measurable in s, x, t (with respect to $\mathfrak{B}_I \times \mathfrak{B} \times \mathfrak{B}_I$).

(S_2) Whatever $s \in I$, $x \in E$, $\Gamma \in \mathfrak{B}$, a random variable τ independent of the future and the s-past, and an $\mathfrak{M}_{s,\tau}$-measurable function $\eta(\omega) \ge \tau(\omega)$ (with values in I) may be, for almost all $\omega \in \Omega$ (in the sense of $P_{s,x}$) the relation

$$P_{s,x}\{x_\eta \in \Gamma \mid x_u,\ s \le u \le \tau\} = P(\tau, x_\tau; \eta, \Gamma)$$

holds.

Obviously, every process that is strongly Markov in the second sense is also strongly Markov in the first sense.

Theorem 1. Let $x(t, \omega)$ be a process strongly Markov in the first sense. Let $\tau \le t$ be a random variable independent of the future and the s-past; let $\xi(\omega)$ be a function measurable with respect to $\mathfrak{M}^t$ and such that

$$M_{s,x}\xi = \int_\Omega \xi(\omega) P_{s,x}(d\omega)$$

exists. Then, for almost all ω (in the sense of $P_{s,x}$),

$$M_{s,x}\{\xi \mid x_u,\ s \le u \le \tau\} = M_{\tau, x_\tau}\xi$$

($M_{s,x}\{\cdot \mid x_u,\ s \le u \le \tau\}$ denotes conditional expectation with respect to the σ-algebra $\mathfrak{M}_{s,\tau}$).

Theorem 2. Let $x(t,\omega)$ be a process strongly Markov in the second sense; let $f(x_1, \dots, x_n, \dots)$ be a $\mathfrak{B} \times \cdots \times \mathfrak{B} \times \cdots$-measurable function on the space $E \times \cdots \times E \times \cdots$; let τ be a random variable independent of the future and the s-past; and let $\eta_1, \eta_2, \dots, \eta_n, \dots \ge \tau$ be a sequence of $\mathfrak{M}_{s,\tau}$-measurable ω-functions such that $M_{s,x} f(x_{\eta_1}, \dots, x_{\eta_n}, \dots)$ exists. Then, for almost all $\omega \in \Omega_\tau$,

$$M_{s,x}\{f(x_{\eta_1}, \dots, x_{\eta_n}, \dots) \mid x_u,\ s \le u \le \tau\} = F(\tau, x_\tau; \eta_1, \dots, \eta_n, \dots),$$

where

$$F(u, y;\ v_1, \dots, v_n, \dots) = M_{u,y} f(x_{v_1}, \dots, x_{v_n}, \dots).$$

3. We shall now assume that E is a metric space and that $\mathfrak{B}$ is the σ-algebra generated by the open subsets of E. We shall say that the process $x(t,\omega)$ is right-continuous if, for every $\omega \in \Omega$, $x(t,\omega)$ is a right-continuous function of t.

Theorem 3. A right-continuous Markov process is strongly Markov in the first sense if and only if it is strongly Markov in the second sense.

Thus, for right-continuous processes one need not distinguish between strongly Markov processes in the first and in the second sense, and one may speak simply of strongly Markov processes.

Theorem 4. Let $x(t,\omega)$ $(0 \le t < \infty,\ \omega \in \Omega)$ be a right-continuous Markov process satisfying condition (J_2). In order that such a process be strongly Markov, it is sufficient that condition (S_2) hold for $\eta = \tau + h$, where h is an arbitrary nonnegative constant.

Introduce the following conditions:

(F_1) Whatever continuous bounded function $f(y)$ $(y \in E)$ may be, the function

$$F(u, y) = \int_E P(u, y; t, dz) f(z)$$

satisfies the relation

$$\lim_{\substack{y \to x \\ u \downarrow s}} F(u, y) = F(s, x)$$

at all points $s \in I,\ x \in E$.

(F_2) Whatever continuous bounded function $f(y)$ $(y \in E)$ is chosen, the function

$$\varphi(u, y) = \int_E P(u, y;\ u + h, dz) f(z)$$

satisfies the relation

$$\lim_{\substack{y \to x \\ u \downarrow s}} \varphi(u, y) = \varphi(s, x)$$

at all points $s \in I,\ x \in E$.

Theorem 5. If a Markov process is continuous from the right and satisfies conditions (J_1)–(F_1) or (J_2)–(F_2), then it is strictly Markov.

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CITED LITERATURE

[1] E. B. Dynkin, A. A. Yushkevich, *Theory of Probability*, **1**, 149 (1956).
[2] J. A. Hunt, Trans. Am. Math. Soc., **81**, 2, 294 (1956).
[3] D. Ray, Trans. Am. Math. Soc., **82**, 2 (1956).

Note: Figure translations are in progress. See original paper for figures.