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Abstract

Full Text

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A NEW METHOD FOR THE APPROXIMATE COMPUTATION OF EIGENVALUES OF THE STURM-LIOUVILLE PROBLEM

(Presented by Academician A. A. Dorodnitsyn on 8 IV 1957)

We solve the question of computing the eigenvalues of the Sturm–Liouville problem

$$-u'' + p(x)u = \lambda u; \qquad u(0) = u(\pi) = 0. \tag{1}$$

1. The following method is known. Let $G(x, y)$ be the Green' s function. Then the relations are valid (where λ_n are the eigenvalues)

$$\sum \lambda_n^{-1} = \int_0^\pi G(x,x)\,dx,$$

$$\sum \lambda_n^{-2} = \int_0^\pi \int_0^\pi G(x,y)G(y,x)\,dy\,dx, \tag{2}$$

$$\dots\dots\dots\dots\dots\dots$$

(trace formulas for powers of the integral operator inverse to the differential one). In the equalities (2) one must replace all eigenvalues, beginning with some one, by their asymptotic values; then for the first eigenvalues one obtains a system of algebraic equations. This method is readily generalized to the more complicated Sturm–Liouville problem $-u'' + pu = \lambda\rho u$ (see, for example, the article by A. A. Dorodnitsyn[5]). Despite its obvious simplicity, the method has an essential drawback: the Green' s function can be written explicitly only in a few cases.

2. It turns out to be possible, instead of sums of negative powers of eigenvalues (2), to compute sums of positive powers $\sum \lambda_n$, $\sum \lambda_n^2, \dots$ (traces of powers of the differential operator). Of course, the divergent series must then be "regularized," i.e., a certain expression must be subtracted from each term so that the series becomes convergent. The first formula of this type (for $\sum \lambda_n$) was first obtained by I. M. Gelfand and B. M. Levitan[1] (see also [3]); the remaining formulas are in works [2,4]. In the present article we use the results of our work [4] in order to show how, with their help,

eigenvalues can be computed. The possibility of computing eigenvalues in this way was pointed out by I. M. Gelfand.

3. The Sturm-Liouville problem (1) is considered. It is convenient from the very beginning to replace the function $p(x)$ by a segment of its Fourier cosine series

$$\sum_{n=0}^{N} a_n \cos nx,$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi p(t)\cos nt\,dt \quad \text{for } n > 0$$

and

$$a_0 = \frac{1}{\pi}\int_0^\pi p(t)\,dt.$$

From such a replacement the first eigenvalues change by an arbitrarily small amount for sufficiently large N. Moreover, if a constant is subtracted from the function $p(x)$, then all eigenvalues also decrease by this constant.

Therefore, for simplicity of the formulas one may assume that $a_0 = 0$. In what follows we shall suppose that

$$p(x) = \sum_{n=1}^{N} a_n \cos nx.$$

As is known, the eigenvalues with large indices admit the asymptotic expansion

$$\lambda_n \sim n^2 + c_0 + \frac{c_2}{n^2} + \frac{c_4}{n^4} + \frac{c_6}{n^6} + \cdots. \tag{3}$$

The coefficients may be found from the formulas of paper (4) or by some other method. For functions $p(x)$ of the form under consideration (all odd derivatives are equal to zero at the endpoints of the interval and

$$\int_0^\pi p(x)\,dx = 0$$

) these coefficients are equal to

$$c_0 = 0;$$

$$c_2 = \frac{1}{4\pi}\int_0^\pi p^2(x)\,dx;$$

$$c_4 = \frac{1}{8\pi}\int_0^\pi p^3(x)\,dx + \frac{1}{16\pi}\int_0^\pi [p'(x)]^2\,dx;$$

$$c_6 = -\frac{5}{4}c_2^2 + \frac{5}{64\pi}\int_0^\pi p^4(x)\,dx + \frac{5}{32\pi}\int_0^\pi p(x)[p'(x)]^2\,dx + \frac{1}{64\pi}\int_0^\pi [p''(x)]^2\,dx. \tag{4}$$

4. As was shown in paper (4), the eigenvalues are connected with the function $p(x)$ by the identities

$$\sum_{n=1}^{\infty}(\lambda_n - n^2) = \frac{p(0)+p(\pi)}{4};$$

$$\sum_{n=1}^{\infty}(\lambda_n^2 - n^4 - 2c_2) = c_2 - \frac{p^2(0)+p^2(\pi)}{5} + \frac{p''(0)+p''(\pi)}{8}; \tag{5}$$

$$\sum_{n=1}^{\infty}(\lambda_n^3 - n^6 - 3c_2n^2 - 3c_4) =$$

$$= \frac{3}{2}c_4 - \frac{p^3(0)+p^3(\pi)}{4} - \frac{3[p^{\mathrm{IV}}(0)+p^{\mathrm{IV}}(\pi)]}{64} + \frac{3[p(0)p''(\pi)+p(\pi)p''(\pi)]}{8}.$$

One could write, in addition to these, further identities for arbitrary powers of λ_n. The series on the left-hand sides of these identities converge, since their terms are of order $O\left(\frac{1}{n^2}\right)$ for large n, as follows from the asymptotic representation (3). However, this convergence is not sufficiently rapid. It can be improved by taking into account that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}.$$

From each term of the series in the left-hand side of the first identity we subtract

$$\frac{c_2}{n^2} + \frac{c_4}{n^4} + \frac{c_6}{n^6},$$

and from the right-hand side,

$$c_2\frac{\pi^2}{6}+c_4\frac{\pi^4}{90}+c_6\frac{\pi^6}{945};$$

from the second identity, respectively,

$$\frac{2c_4}{n^2}+\frac{c_2^2+2c_6}{n^4}$$

and

$$c_4\frac{\pi^2}{3}+(c_2^2+2c_6)\frac{\pi^4}{90};$$

from the third identity, respectively,

$$\frac{3c_2^2+3c_6}{n^2}$$

and

$$(3c_2^2+3c_6)\frac{\pi^2}{6}.$$

Now the terms of the first series are quantities $O\left(\frac{1}{n^8}\right)$, of the second $O\left(\frac{1}{n^6}\right)$, and of the third $O\left(\frac{1}{n^4}\right)$. The first series converges fastest of all, the last slowest of all. This is precisely what we need,

since the sum of the first eigenvalues must be known with a smaller absolute error than the sum of their squares and, still more, cubes. The relative errors turn out to be approximately equal.

5. From identities (5) one can obtain an approximate method for computing the eigenvalues λ_n. The first approximation is obtained if we restrict ourselves to the first identity and assume that all eigenvalues, beginning with the second, are quite accurately equal to the asymptotic values n^2. Then all terms of the series beginning with the second term may be neglected:

$$\lambda_1 \approx 1-\frac{p(0)+p(\pi)}{4}. \tag{6'}$$

Without writing out the second approximation, which is obtained if in two identities all terms beginning with the third are neglected, let us immediately write the third approximation. For this we take three identities (with improved convergence) and in them discard all terms beginning with the fourth:

$$\lambda_1+\lambda_2+\lambda_3 \approx 14-\frac{p(0)+p(\pi)}{4}-c_2\cdot 0.2838230-c_4\cdot 0.007478-c_6\cdot 0.000346;$$

$$\lambda_1^2+\lambda_2^2+\lambda_3^2 \approx 98+7c_2-\frac{p^2(0)+p^2(\pi)}{4}+\frac{p''(0)+p''(\pi)}{8}-$$

$$-c_4\cdot 0.567646-(c_2^2+2c_6)\cdot 0.007478; \qquad (6")$$

$$\lambda_1^3+\lambda_2^3+\lambda_3^3 \approx 794+42c_2+10.5c_4-\frac{p^3(0)+p^3(\pi)}{4}-\frac{3[p^{\mathrm{IV}}(0)+p^{\mathrm{IV}}(\pi)]}{64}+$$

$$+\frac{3[p(0)p''(0)+p(\pi)p''(\pi)]}{8}-(3c_6+3c_2^2)\cdot 0.283823.$$

It remains to solve a system of three equations with three unknowns. If necessary, the accuracy can be increased by writing the following identities (according to the rules of paper (4)).

6. As an example, let us consider the Mathieu equation

$$-u''+\cos 2x\cdot u=\lambda u; \qquad u(0)=u(\pi)=0.$$

By the rough formula (6′) one may obtain $\lambda_1 \approx 0.5$. Let us compute c_2, c_4, c_6 by formulas (4):

$$c_2=0.125; \qquad c_4=0.125; \qquad c_6=0.1348\ldots$$

Let us form the system of equations (6″)

$$\lambda_1+\lambda_2+\lambda_3 \approx 13.46354;$$

$$\lambda_1^2+\lambda_2^2+\lambda_3^2 \approx 97.3019;$$

$$\lambda_1^3+\lambda_2^3+\lambda_3^3 \approx 795.434.$$

From this system one can without difficulty obtain the cubic equation satisfied by $\lambda_1, \lambda_2, \lambda_3$:

$$\lambda^3 - 13.46354\,\lambda^2 + 41.9825\,\lambda - 16.8797 = 0.$$

We solve the equation by Newton' s method, taking as initial approximations the asymptotic values 0.5; 4; 9. We find:

$$\lambda_1 \approx 0.47061; \qquad \lambda_2 \approx 3.97929; \qquad \lambda_3 \approx 9.01364.$$

The exact values, obtained from the formulas in the book (6), are

$$\lambda_1 = 0.47065\ldots; \qquad \lambda_2 = 3.97919\ldots; \qquad \lambda_3 = 9.01371\ldots$$

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Note: Figure translations are in progress. See original paper for figures.