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Consider singular integral equations of the form

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Abstract

Full Text

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SOLUTION OF A CLASS OF SINGULAR INTEGRAL EQUATIONS WITH A CONTOUR INTEGRAL

(Presented by Academician S. L. Sobolev, 5 X 1956)

§ 1.

Consider singular integral equations of the form

$$\psi(s,t)=\lambda\int_0^t d\tau\int_C K_0(r^2_{pp_1},t-\tau)\psi(s_1,\tau)\,ds_1+f(s,t)\qquad(t>0),\tag{1}$$

where r_{pp_1} is the distance between the points p and p_1 with coordinates s and s_1 in the arc-coordinate system,

$$K_0(r^2_{pp_1},t-\tau)=\frac{1}{(t-\tau)^{3/2}}\int_0^\infty\rho(z)\left[1-\frac{r^2_{pp_1}}{2a^2(z)(t-\tau)}\right]\exp\left[-\frac{r^2_{pp_1}}{4a^2(z)(t-\tau)}\right]dz,\tag{2}$$

$$\rho(z)=(z^2+a_1^2)^{-3/2}(z^2+a_2^2)^{-1/2},\qquad a^2(z)=a_2^2(z^2+a_1^2)/(z^2+a_2^2).\tag{3}$$

In this equation the function $f(s,t)$ has a derivative of bounded variation, and

$$t^\sigma|f(s,t)|,\ V(t^\sigma\partial f(s,t)/\partial s)\le M\qquad(0\le\sigma<1).\tag{4}$$

In the present paper we shall show that equation (1), for arbitrary λ, has no solution in the class of functions satisfying the inequality

$$|\psi(s_1,t)-\psi(s_2,t)|\le Mt^{-\sigma}|s_1-s_2|^\alpha\tag{5}$$

(σ and α independently satisfy the inequalities $0\le\sigma<1$, $0<\alpha\le1$), and exists only for $\lambda<\lambda_0$, where λ_0 is a completely definite number. Consequently, on the basis of Abel' s theorem, the method of successive approximations cannot be applied to equation (1), since this method does not give a complete solution.

§ 2.

We investigate equation (1). In view of the periodicity of the functions r_{pp_1} and $f(s,t)$ with respect to s, with period $2l$ (the length of the arc of the curve C), the required function $\psi(s,t)$ must be periodic with period $2l$. Therefore we represent the solution in the form of a Fourier series, if it exists. If the Fourier series of the function $\psi(s,t)$ diverges, then a solution of equation (1) cannot exist in the class of functions satisfying inequality (5). Let

$$\psi(s,t) = \frac{\psi_0(s,t)}{2} + \sum_{k=1}^{\infty} \left(\psi_k^{(1)}(t) \cos \frac{k\pi s}{l} + \psi_k^{(2)}(t) \sin \frac{k\pi s}{l} \right). \tag{6}$$

On the basis of (4), the function $f(s,t)$ is also expanded in a Fourier series

$$f(s,t) = \frac{f_0(t)}{2} + \sum_{k=1}^{\infty} \left(f_k^{(1)}(t) \cos \frac{k\pi s}{l} + f_k^{(2)}(t) \sin \frac{k\pi s}{l} \right), \tag{7}$$

moreover

$$\left| f_k^{(i)}(t) \right| \le M/k^2 t^\sigma \quad (i = 1, 2). \tag{8}$$

It is easy to show that, if the curve C is sufficiently smooth, then

$$r_{pp_1}^2 = (s - s_1)^2 \varphi(s, s_1), \tag{9}$$

where $\varphi(s, s_1)$ is a known function depending on the form of the curve C. Multiply the function $\frac{1}{l} \cos \frac{n\pi s}{l}$ by both sides of equation (1) and integrate along the contour C. Then

$$\psi_n^{(1)}(t) = \lambda \int_0^t d\tau \int_C \left\{ \frac{1}{l} \int_C K_0((s - s_1)^2 \varphi(s, s_1), t - \tau) \cos \frac{n\pi s}{l} \, ds \right\} \psi(s_1, \tau) \, ds_1 + f_n^{(1)}(t).$$

Making the substitution $s - s_1 = s'$, we obtain

$$\begin{aligned} \psi_n^{(1)}(t) = \lambda \int_0^t d\tau \int_C \left\{ \frac{1}{l} \int_{-l}^{+l} K_0(s'^2 \varphi(s' + s_1, s_1), t - \tau) \cos \frac{n\pi s'}{l} \, ds' \right\} \psi(s_1, \tau) \cos \frac{n\pi s_1}{l} \, ds_1 \\ + \lambda \int_0^t d\tau \int_C \left\{ \frac{1}{l} \int_{-l}^{+l} K_0(s'^2 \varphi(s' + s_1, s_1), t - \tau) \sin \frac{n\pi s'}{l} \, ds' \right\} \psi(s_1, \tau) \sin \frac{n\pi s_1}{l} \, ds_1 \\ + f_n^{(1)}(t). \end{aligned} \tag{10}$$

The inner integrals are not computed in explicit form. Therefore we shall separate the principal parts from these integrals, and estimate the remainder. As a result we shall have

$$\psi_n^{(1)}(t) = \lambda \int_0^t \left\{ 4\sqrt{\pi}\lambda_n \int_0^\infty \rho(z) a^3(z) \exp\left[-\lambda_n^2 a^2(z)(t-\tau)\right] \, dz \right\} \psi_n^{(1)}(\tau)\, d\tau + \int_0^t d\tau \int_C \Phi_n^{(1)}(s_1, t-\tau) \psi(s_1, \tau)\, ds_1 \tag{11}$$

Entirely analogously, we obtain:

$$\psi_n^{(2)}(t) = \lambda \int_0^t \left\{ 4\sqrt{\pi}\lambda_n \int_0^\infty \rho(z) a^3(z) \exp\left[-\lambda_n^2 a^2(z)(t-\tau)\right] \, dz \right\} \psi_n^{(2)}(\tau)\, d\tau + \int_0^t d\tau \int_C \Phi_n^{(2)}(s_1, t-\tau) \psi(s_1, \tau)\, ds_1 \tag{12}$$

where $\lambda_n = n\pi/l$, and $\Phi_n^{(i)}(s_1, t-\tau)$ satisfy the inequality

$$\left| \Phi_n^{(i)}(s_1, t-\tau) \right| \leq M n^{-3/2} (t-\tau)^{-3/4} \quad (i = 1, 2). \tag{13}$$

Equations (11) and (12) shall be called the equations associated with the singular equation (1).

§ 3. We now consider the integral equation

$$\psi_n(t) = \lambda \int_0^t K_n^0(t-\tau) \psi_n(\tau)\, d\tau + f_n(t), \tag{14}$$

where

$$K_n^{(0)}(t-\tau) = 4\sqrt{\pi}\lambda_n \int_0^\infty \rho(z) a^3(z) \exp\left[-\lambda_n^2 a^2(z)(t-\tau)\right] \, dz. \tag{15}$$

Equation (14) is obtained from equations (11) and (12) by dropping the second terms. We shall call it the **characteristic equation of the adjoint equations**.

Applying the operational method to equation (14), we obtain the solution in explicit form

$$\psi_n(t) = f_n(t) + \lambda \int_0^t \Gamma_n(t-\tau; \lambda) f_n(\tau)\, d\tau, \tag{16}$$

where

$$\Gamma_n(t-\tau;\lambda) = \frac{2\pi^{3/2}(|\nu|-\nu)}{(\nu^2-1)(a_1^2\nu^2-a_2^2)}\lambda_n^2 \exp[-\lambda_n^2 d_0(t-\tau)]+$$

$$+\,4\sqrt{\pi}\,a_2^3\mu^2\lambda_n \int_0^\infty \frac{z^2}{(z^2+a_2^2)^2(\nu^2 z^2+a_2^2)} \exp[-\lambda_n^2 d^2(z)(t-\tau)]\,dz, \qquad (17)$$

$$d_0 = (a_1^2\nu^2 - a_2^2)/(\nu^2-1), \qquad \nu = \mu - 1, \qquad \mu = (a_1^2 - a_2^2)/2\pi^{-3/2}\lambda. \qquad (18)$$

It is obvious that our solution is meaningful if $\nu \neq \pm 1$, $\nu \neq \pm a_2/a_1$, but by direct verification one can establish that for $\nu = 1$, $\nu = \pm a_2/a_1$ the solution can be obtained from (16) by a limiting passage; if, however, $\nu = -1$, then it follows from (18) that $\lambda = \infty$. Therefore the latter case is excluded from consideration.

It is easy to verify that the resolvent $\Gamma_n(t-\tau;\lambda)$ satisfies the equations:

$$\Gamma_n(t-\tau;\lambda) = K_n^0(t-\tau) + \lambda \int_\tau^t \Gamma_n(t-t_1;\lambda)K_n^0(t_1-\tau)\,dt_1, \qquad (19)$$

$$\Gamma_n(t-\tau;\lambda) = K_n^0(t-\tau) + \lambda \int_0^t K_n^0(t-t_1)\Gamma_n(t_1-\tau;\lambda)\,dt_1; \qquad (20)$$

For what follows we introduce the operators

$$B_n\psi(t) = \psi(t) - \lambda \int_0^t K_n^0(t-\tau)\psi(\tau)\,d\tau, \qquad (21)$$

$$B^{-1}\psi(t) = \psi(t) + \lambda \int_0^t \Gamma_n(t-\tau;\lambda)\psi(\tau)\,d\tau. \qquad (22)$$

On the basis of formulas (19) and (20), these operators have the following properties:

$$BB^{-1}\psi(t) = B^{-1}B\psi(t) = \psi(t). \qquad (23)$$

It follows directly from these equalities that the equations

$$B_n\psi(t) = 0, \qquad B_n^{-1}\psi(t) = 0 \qquad (24)$$

have only the trivial solution. Consequently, equalities (14) and (16) are equivalent.

§ 4. We rewrite equations (11) and (12) with the aid of the operators (21) and (22):

$$B_n\psi_n^{(1)}(t) = \int_0^t d\tau \int_C \Phi_n^{(1)}(s_1, t-\tau)\psi(s_1,\tau)\,ds_1 + f_n^{(1)}(t), \tag{11_1}$$

$$B_n\psi_n^{(2)}(t) = \int_0^t d\tau \int_C \Phi_n^{(2)}(s_1, t-\tau)\psi(s_1,\tau)\,ds_1 + f_n^{(2)}(t). \tag{12_1}$$

Applying the operator B_n^{-1} to these equalities, on the basis of (23) we obtain

$$\psi_n^{(1)}(t) = \int_0^t d\tau \int_C B_n^{-1}\Phi_n^{(1)}(s_1, t-\tau)\psi(s_1,\tau)\,ds_1 + B_n^{-1}f_n^{(1)}(t), \tag{25}$$

$$\psi_n^{(2)}(t) = \int_0^t d\tau \int_C B_n^{-1}\Phi_n^{(2)}(s_1, t-\tau)\psi(s_1,\tau)\,ds_1 + B_n^{-1}f_n^{(2)}(t). \tag{26}$$

It is clear that these equalities are equivalent to equalities (11) and (12). Substituting (25) and (26) into the series (6) and assuming that the resulting series converges uniformly, we obtain

$$\psi(s,t) = \int_0^t d\tau \int_C K(s, s_1, t-\tau)\psi(s_1,\tau)\,ds_1 + f_1(s,t), \tag{27}$$

where

$$K(s, s_1, t-\tau) = \frac{1}{2}\Phi_0(t) + \sum_{n=1}^{\infty}\left[B_n^{-1}\Phi_n^{(1)}(s_1, t-\tau)\cos\frac{n\pi s}{l} + B_n^{-1}\Phi_n^{(2)}(s_1, t-\tau)\sin\frac{n\pi s}{l}\right], \tag{28}$$

$$f_1(s,t) = \frac{1}{2}f_0(t) + \sum_{n=1}^{\infty}\left[B_n^{-1}f_n^{(1)}(t)\cos\frac{n\pi s}{l} + B_n^{-1}f_n^{(2)}(t)\sin\frac{n\pi s}{l}\right]. \tag{29}$$

If $d_0 \leq 0$ and $\nu < 0$, then the operator B_n^{-1} grows exponentially, or as n^2 as n increases, and in this case the series (28) and (29) diverge. In order for these series to converge uniformly, it is necessary and sufficient that $d_0 > 0$ or $\nu \geq 0$. Translating these conditions into λ, we have:

$$\lambda < a_1(a_1 + a_2)/2\pi^{3/2} = \lambda_0. \tag{30}$$

On the basis of (30) and (13),

$$|K(s, s_1, t-\tau)| \leq M_1 (t-\tau)^{-3/4}, \tag{31}$$

$$|K(s', s_1, t-\tau) - K(s'', s_1, t-\tau)| \leq M_1 (t-\tau)^{-3/4} |s' - s''|^{\alpha} \quad (0 < \alpha < 1/2), \tag{32}$$

and on the basis of (8)

$$|f_1(s, t)| \leq M_2 t^{-\sigma}, \quad |f_1(s', t) - f_1(s'', t)| \leq M_2 t^{-\sigma} |s' - s''|^{\alpha_1} \quad (0 < \alpha_1 < 1). \tag{33}$$

The integral equation (27) can now be integrated by the method of successive approximations, and its solution will satisfy condition (5).

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Note: Figure translations are in progress. See original paper for figures.

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