The functions

1957

SovietRxiv

Abstract

Full Text

Consider the differential operators $(m)u = \dfrac{\partial}{\partial x} M u$, $(\overline{m})u = \dfrac{\partial}{\partial y}\left(\dfrac{u}{M}\right)$. Let $\mathfrak{M}$ be the set of nonnegative integral powers of the operator m; let $\overline{\mathfrak{M}}$ be the set of nonnegative integral powers of the operator $(\overline{m})$. The elements of these sets (of either one—indifferently) will be denoted by the letters $a_1, a_2, \ldots$; the exponent a_k will be called the exponent of the power (m) or $(\overline{m})$ that coincides with a_k.

Introduce $\otimes$-multiplication for the a_k, subject to the following rules of calculation. An expression placed in square brackets is subjected to the ordinary action of a differential operator.

1) $(a_k)u = (a_k)[u]$.
2) If $a_n \in \mathfrak{M},\ a_{n-1} \in \mathfrak{M}$, or $a_n \in \overline{\mathfrak{M}},\ a_{n-1} \in \overline{\mathfrak{M}}$, then

$$(a_n \otimes a_{n-1} \otimes \cdots \otimes a_1)u = (a_n)[(a_{n-1} \otimes \cdots \otimes a_1)u].$$

3) If $a_n \in \mathfrak{M},\ a_{n-1} \in \overline{\mathfrak{M}}$, then

$$(a_n \otimes a_{n-1} \otimes \cdots \otimes a_1)u = a_n[M^s(a_{n-1} \otimes \cdots \otimes a_1)u],$$

and if $a_n \in \overline{\mathfrak{M}},\ a_{n-1} \in \mathfrak{M}$, then

$$(a_n \otimes a_{n-1} \otimes \cdots \otimes a_1)u = a_n[M^{-s}(a_{n-1} \otimes \cdots \otimes a_1)u],$$

where k_i are the exponents of a_i and $s = \sum_{i<n} k_i$.

It is known that $P = \dfrac{\partial^2}{\partial x\,\partial y} \ln M$ is, up to a constant factor, the only integral invariant $(1,1)$. It is also known that any integral invariant of order r can be represented as a polynomial with real coefficients in invariants of the form $(m^\rho \otimes \overline{m}^\sigma)P$ and of orders not exceeding the order of the given invariant.

A necessary and sufficient condition for the nomographability of the equation $z = \varphi(x, y)$ in a domain G, as is known (2), is the compatibility in G of the Gronwall differential equations

$$\begin{gathered}
\frac{\partial}{\partial x}\left(2\frac{\partial \overline{C}}{\partial y} + \frac{\partial \overline{D}}{\partial x}\right) - \overline{C}\left(2\frac{\partial \overline{C}}{\partial y} + \frac{\partial \overline{D}}{\partial x}\right) = 0, \\
\frac{\partial}{\partial y}\left(\frac{\partial \overline{C}}{\partial y} + 2\frac{\partial \overline{D}}{\partial x}\right) - \overline{D}\left(\frac{\partial \overline{C}}{\partial y} + 2\frac{\partial \overline{D}}{\partial x}\right) = 0, \\
\overline{D} = M\overline{C} + N,
\end{gathered} \tag{3}$$

where $\overline{C}$ is the Gronwall function; $\overline{D}$ is its complement; M, N are expressed by the formulas

$$M = -\frac{\varphi_y}{\varphi_x}, \qquad N = \frac{\partial M}{\partial x} + \frac{1}{M}\frac{\partial M}{\partial y}. \tag{4}$$

The functions

$$C = \overline{C} + \frac{\partial}{\partial x}\ln M, \qquad D = \overline{D} - \frac{\partial}{\partial y}\ln M \tag{5}$$

will be called, respectively, the principal parts of the Gronwall function and of its complement. Obviously, $D = MC$.

The functions adjoint to the Gronwall function will be called

$$z_1 = \frac{1}{3}\left(2\frac{\partial \overline{C}}{\partial y} + \frac{\partial \overline{D}}{\partial x}\right), \qquad z_2 = \frac{1}{3}\left(\frac{\partial \overline{C}}{\partial y} + 2\frac{\partial \overline{D}}{\partial x}\right). \tag{6}$$

The Gronwall differential equations can be represented in the form

$$\begin{gathered} (m)z_1 = MCz_1, \qquad (\overline{m})z_2 = Cz_2, \\ (m)C = 2z_2 - z_1 - P, \qquad (m)MC = 2z_1 - z_2 + P, \end{gathered} \tag{7}$$

where $P = \dfrac{d^2}{dx\,dy}\ln M$ is the already mentioned Saint-Robert invariant, up to a constant factor.

up to an inessential factor, coinciding with the Blaschke curvature net (^{1}).

The coefficients of the polynomial $\sigma(C, z_1, z_2)$ in C, z_1, z_2 contain the homographic invariants $T = (1,3)$, $W = (1,4)$, expressed by the formulas

$$T = \sum_{\substack{i+k=2 \\ i\geq 0,\ k\geq 0}} (m^i \otimes \overline{m}^k)P - 3MP^2, \tag{8}$$

$$W = (m \otimes \overline{m} \otimes m)P + M^3(\overline{m} \otimes m \otimes \overline{m})P + 7MP(m)P + M^2P(\overline{m})P,$$

the polynomial $\sigma(C, z_1, z_2)$ has the form

$$\sigma(C, z_1, z_2) = M^3PC^3 + M\{8MP(z_1 - z_2) - T\}C - \{6M^2z_1(\overline{m})P - 6Mz_2(m)\}P + W. \tag{9}$$

Theorem 1. The Gronwall system of differential equations, under the condition of sufficient smoothness of $\varphi(x, y)$, always admits the prolongation

$$8MP(m)z_2 = \sigma(C, z_1, z_2), \tag{10}$$

$$(\overline{m})z_1 = \frac{1}{M}(m)z_2 + (2z_2 - 2z_1 - P)C - \frac{1}{M}(m)P - (\overline{m})P. \tag{11}$$

Proof. Using the identities expressing the commutation rules in double and triple products of the operators (m) and $(\overline{m})$,

$$M^2(\overline{m} \otimes m)u - (m \otimes \overline{m})u = 2MPu, \tag{12^1}$$

$$M^3(\overline{m}^2 \otimes m)u - (m \otimes \overline{m}^2)u = 5M^2P(\overline{m})u + 2M^2(\overline{m})P, \tag{12^2}$$

one forms and compares $(m \otimes \overline{m})C$ and $(\overline{m} \otimes m)C$, $(\overline{m}^2 \otimes m)z_1$ and $(m \otimes \overline{m}^2)z_1$. It can be shown that equations (10), (11) exhaust all independent equations relating C, z_1, z_2 and their first derivatives, which follow from (7) and from the commutation rules in triple products of the operators (m) and $(\overline{m})$. For example, comparing $(\overline{m} \otimes m^2)z_2$ and $(m^2 \otimes \overline{m})z_2$, one obtains the following consequence of equations (10), (11):

$$-\frac{8}{M}P(\overline{m})z_1 = \sigma_1(C, z_1, z_2), \tag{13}$$

where $\sigma_1(C, z_1, z_2)$ is symmetric to $\sigma(C, z_1, z_2)$, i.e. it is obtained from the polynomial $\sigma(C, z_1, z_2)$ by interchanging x, y.

Theorem 2. If C, z_1, z_2 is a solution of the Gronwall system of differential equations and in the domain G the Saint-Robert invariant P is locally different from the identically zero invariant, then C, z_1, z_2 satisfy the system of algebraic equations

$$\Phi_1(C, z_1, z_2) = 3MC^3(m)P - 20MP(z_1 - 2z_2 + P)C^2 + 32P(z_1^2 + 2z_1z_2 - 2z_2^2) + \cdots = 0, \tag{14}$$

$$\Phi_2(C, z_1, z_2) = 3M^2C^3(\overline{m})P + 20MP(2z_1 - z_2 + P)C^2 + 32P(z_2^2 + 2z_1z_2 - 2z_1^2) + \cdots = 0$$

(the dots indicate omitted lower-order terms, whose coefficients are homographic invariants (^{6})).

Proof. Equations (7), (10), (11) can be solved with respect to all $(\overline{m})C, \dots, (\overline{m})z_2$. Forming the necessary integrability conditions, for which one has to use the commutation rule (12^1),

we find what is required. Note that one of the three relations obtained in this way is a trivial identity.

Theorem 3. *If, in the domain G, the Saint-Robert invariant P does not locally vanish identically, then the principal part $C(x, y)$ of the Gronwall function is found as a common root of a system of polynomials whose coefficients are nomographic invariants.*

Theorem 4. *The necessary and sufficient conditions for the nomographability of the equation $z = \varphi(x, y)$ with a sufficiently smooth and monotone right-hand side reduce to two, generally speaking independent, relations connecting the partial derivatives, of order not higher than 9, of the function $\varphi(x, y)$.*

Proof of Theorems 3 and 4. If P in the domain G is locally different from identical zero, then equations (14) can be solved with respect to z_1, z_2, which gives an expression for the adjoint functions in terms of C, x, y. Substituting these expressions for z_1, z_2 into the differential equations (7), we find what is required. For $P \equiv 0$, as is known ([2]), the equation $z = \varphi(x, y)$ is always nomographable.

Using invariants, one can in fact write down algebraic equations with respect to C and the conditions for nomographability; moreover, both the equations for C and the conditions for nomographability are simplified if it is assumed in advance that one or two scales of the nomogram are rectilinear ([8,9]).

Ivanovo State Pedagogical Institute

Received
5 IV 1957

CITED LITERATURE

1. W. Blaschke, *Einführung in die Geometrie der Waben*, 1955.
2. T. H. Gronwall, J. de Math., 6 sér., 8, 59 (1912).
3. M. A. Kreines, N. D. Aizenshtat, Matem. sborn., 37 (79), 337 (1955).
4. M. A. Kreines, N. D. Aizenshtat, DAN, 95, 1137 (1954).
5. S. V. Smirnov, DAN, 69, 99 (1949).
6. S. V. Smirnov, Uchen. zap. Ivanovsk. ped. inst., 4, 22 (1953).
7. S. V. Smirnov, Tr. 3rd All-Union Mathematical Congress, 2, 142 (1956).
8. S. V. Smirnov, Uspekhi matem. nauk, 11, issue 4 (70), 177 (1956).
9. S. V. Smirnov, Uchen. zap. Ivanovsk. ped. inst., ser. matem., issue 5 (1957).

10. E. T. Smorkachev, DAN, 113, No. 4 (1957).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.