MATHEMATICS

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Abstract

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ON THE QUESTION OF REPRESENTATION OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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1. Let there be two separable Hilbert spaces with reproducing kernels ([1]), consisting respectively of functions $\{f(z)\}$ $(z \in G)$ and $\{\varphi(\zeta)\}$ $(\zeta \in \mathcal{G})$, where G and $\mathcal{G}$ are certain sets. Denote by (,) and [,], respectively, the scalar products in H and $\mathcal{H}$. If $\{p_n(z)\}_1^\infty$ and $\{q_n(z)\}_1^\infty$ are orthonormal bases respectively in H and $\mathcal{H}$, then, as is known, $\sum_1^\infty |p_n(z)|^2 < \infty$ $(z \in G)$ and $\sum_1^\infty |q_n(\zeta)|^2 < \infty$ $(\zeta \in \mathcal{G})$, and the reproducing kernels $g(z_1, z_2)$ and $h(\zeta_1, \zeta_2)$ of these spaces are expressed through them in the form

$$g(z_1, z_2) = \sum_1^\infty p_n(z_1)\overline{p_n(z_2)}$$

and

$$h(\zeta_1, \zeta_2) = \sum_1^\infty q_n(\zeta_1)\overline{q_n(\zeta_2)} \quad (z_1, z_2 \in G;\ \zeta_1, \zeta_2 \in \mathcal{G}).$$

We now construct the kernel

$$k(z, \zeta) = \sum_1^\infty p_n(z)\overline{q_n(\zeta)} \quad (z \in G,\ \zeta \in \mathcal{G}). \tag{1}$$

Obviously, $k(z, \zeta) \subset H$ $(\zeta \in \mathcal{G})$ and $\overline{k(z, \zeta)} \subset \mathcal{H}$ $(z \in G)$.

Theorem 1. By the relations

$$f(z) = [\varphi(\zeta), \overline{k(z, \zeta)}], \qquad \varphi(\zeta) = (f(z), k(z, \zeta)) \tag{2}$$

the kernel $k(z, \zeta)$ defines an isometric operator V, mapping $\mathcal{H}$ onto H; here $f(z)$ and $\varphi(\zeta)$ correspond to each other in this isometry. Conversely, to every isometric operator V mapping $\mathcal{H}$ onto H there corresponds a kernel of the form (1), through which the action of the operator V and of its inverse V^{-1} is written by formulas (2).

The proof of this theorem is obtained directly by expanding the scalar products (2). By formulas (2), to each function

$$\varphi(\zeta) = \sum_{1}^{\infty} \alpha_n q_n(\zeta) \subset \mathcal{H}$$

there is put in correspondence the function

$$f(z) = \sum_{1}^{\infty} \alpha_n p_n(z) \subset H,$$

and conversely. The kernel for a prescribed operator V can be obtained if one chooses $p_n(z) = V q_n(\zeta)$, where $\{q_n(\zeta)\}$ is any orthonormal basis in $\mathcal{H}$.

Let us also note that the kernel $k(z, \zeta)$ is connected with the reproducing kernels of H and $\mathcal{H}$ by the relations

$$g(z_1, z_2) = [\overline{k(z_2, \zeta)}, \overline{k(z_1, \zeta)}] \quad (z_1, z_2 \in G)$$

and

$$h(\zeta_1, \zeta_2) = (k(z, \zeta_2), k(z, \zeta_1)) \quad (\zeta_1, \zeta_2 \in \mathcal{G}).$$

In particular, when $H = \mathcal{H}$ and $(\, , \,) = [\, , \,]$, formulas (2) give the general form of a unitary operator in H ([1,2]).

2. Suppose now that the space $\mathcal{H}$ is embedded in a broader Hilbert space $\widetilde{\mathcal{H}}$, which may fail to possess a reproducing kernel. We shall denote the scalar product in $\widetilde{\mathcal{H}}$ also by $[\, , \,]$. Then any element $\widetilde{\varphi} \subset \widetilde{\mathcal{H}}$ can be decomposed as $\widetilde{\varphi} = \varphi(\zeta) + \psi$, where $\varphi(\zeta) \subset \mathcal{H}$ and $\psi \perp \mathcal{H}$, and

$$[\widetilde{\varphi}, \overline{k(z, \zeta)}] = [\varphi(\zeta) + \psi, \overline{k(z, \zeta)}] = [\varphi(\zeta), \overline{k(z, \zeta)}] = f(z).$$

Thus we have obtained the following generalization of Aronszajn' s embedding theorem (1):

Theorem 2. *If the space $\mathcal{H}$ is embedded in a broader Hilbert space $\widetilde{\mathcal{H}}$, then every element $f(z) \subset H$ is representable in the form*

$$f(z) = [\widetilde{\varphi}, \overline{k(z, \zeta)}] \qquad (\widetilde{\varphi} \subset \widetilde{\mathcal{H}}). \tag{3}$$

Among all elements $\widetilde{\varphi}$ representing one and the same function $f(z) \subset H$, the least norm in $\widetilde{\mathcal{H}}$ is attained by the function $\varphi = \varphi(\zeta) = (f(z), k(z, \zeta))$.

This theorem can be used to obtain various integral representations of classes of analytic functions that form Hilbert spaces with a reproducing kernel. For example, let $H \equiv M(2\sigma, \alpha)$ of Djrbashian (3), $\mathcal{H} \equiv H_2$ of Riesz (4), and let $\widetilde{\mathcal{H}}$ be the set of all $\widetilde{\varphi}(e^{i\theta})$ for which

$$\int_0^{2\pi} |\widetilde{\varphi}(e^{i\theta})|^2 \, d\theta < \infty.$$

In this case,

$$p_n(z) = \sqrt{2\sigma\alpha}\,(2\sigma)^{n/\alpha} \left[\Gamma\left(1 + \frac{2n}{\alpha}\right)\right]^{-1/2} z^n \quad (n = 0, 1, \ldots),$$

$$q_n(\zeta) = \zeta^n \quad (n = 0, 1, \ldots).$$

Taking into account the definition of the scalar products in these spaces, on the basis of Theorem 2 we have: every entire function $f(z)$ of class of growth less than $[\alpha, \sigma]$ is representable in the form

$$f(z) = \sqrt{2\sigma\alpha}\,(2\pi)^{-1} \int_0^{2\pi} \widetilde{\varphi}(e^{i\theta}) \left\{ \sum_0^{\infty} (2\sigma)^{n/\alpha} \left[\Gamma\left(1 + \frac{2n}{\alpha}\right)\right]^{-1/2} (ze^{-i\theta})^n \right\} d\theta,$$

where $\widetilde{\varphi}(e^{i\theta})$ is some function satisfying the condition

$$\|\widetilde{\varphi}\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\widetilde{\varphi}(e^{i\theta})|^2 \, d\theta < \infty.$$

Among all functions $\widetilde{\varphi}$ representing one and the same function $f(z)$, $\min \|\widetilde{\varphi}\|$ is attained by the function

$$\widetilde{\varphi}(e^{i\theta}) = \lim_{r \to 1-0} \sqrt{2\sigma\alpha}\,(2\pi)^{-1} \int_0^{\infty} \int_0^{2\pi} f(\rho e^{i\varphi})\, e^{-2\sigma\rho^{\alpha}} \rho^{\alpha-1} \times$$

$$\times \left\{ \sum_{n=0}^{\infty} (2\sigma)^{n/\alpha} \left[\Gamma\left(1 + \frac{2n}{\alpha}\right)\right]^{-1/2} (\rho r)^n e^{in(\theta - \varphi)} \right\} d\rho \, d\varphi.$$

3. To represent various classes of analytic functions by means of real functions subject to one or another condition of quadratic summability, we introduce the following spaces of harmonic functions in the disk C_R $(|z| < R)$. Let

$$E(z) = \sum_0^{\infty} \alpha_n^{-1} z^n \qquad (0 < \alpha_n < \infty)$$

be analytic in the disk C_R $(0 < R \le \infty)$. Consider the class U_E^2 of all functions $\{u(z)\}$ harmonic in the disk C_R, for which the corresponding···

analytic functions $f_u(z)$ ($\operatorname{Re} f_u(z) = u(z)$) belong to Z_E^2 (5). We introduce in U_E^2 the norm

$$\|u(z)\|_{U_E^2} = \inf_{f_u} \|f_u(z)\|_{Z_E^2}. \tag{4}$$

If $u(z) = a_0 + \sum_1^\infty r^n(a_n \cos n\theta + b_n \sin n\theta)$, then, obviously,

$$\|u(z)\|^2 = d_0^2 a_0^2 + \sum_1^\infty (a_n^2 + b_n^2)\alpha_n.$$

If $\tilde{u}(z) = \tilde{a}_0 + \sum_1^\infty r^n(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta)$ also belongs to U_F^2, then the scalar product consistent with (4) has the form

$$(u(z), \tilde{u}(z)) = a_0 \tilde{a}_0 \alpha_0 + \sum_1^\infty (a_n \tilde{a}_n + b_n \tilde{b}_n)\alpha_n. \tag{5}$$

Thus we turn U_E^2 into a complete separable Hilbert space with the orthonormal basis

$$\frac{1}{\sqrt{\alpha_0}}, \left\{ \frac{r^n \cos n\theta}{\sqrt{\alpha_n}}, \frac{r^n \sin n\theta}{\sqrt{\alpha_n}} \right\}_1^\infty.$$

Taking (5) into account, we obtain Theorem 3.

Theorem 3. *The space U_E^2 possesses the reproducing kernel $F(z\bar{\zeta})$, where $F(z) = \operatorname{Re} E(z)$, i.e. for every function $u(z) \subset U_E^2$ one has*

$$u(\zeta) = (u(z), F(z\bar{\zeta})) \qquad (|\zeta| < R). \tag{6}$$

Let us also note that weak convergence in U_E^2 entails uniform convergence in any circle C_r ($r < R$), while strong convergence implies uniform convergence in C_R with weight $[E(|\zeta|^2)]^{-1/2}$. All properties of the space Z_E^2 can be transferred to these spaces, and on the basis of (6) one can obtain various representations of these classes of functions by embedding U_E^2 in a broader space $\widetilde{U}$, or by using mappings of various spaces of type U_E^2. Formula (6) may be regarded as a generalization of the Poisson formula, which can be obtained from it if one chooses

$$E(z) = \frac{R^2 + z}{2(R^2 - z)}.$$

4. For the representation of analytic functions through real-valued functions we need to find an analogue of Schwarz' s formula (7) for our spaces Z_E^2 and U_E^2. Let

$$f(z) = \sum_0^\infty (a_n + ib_n) z^n = u(z) + iv(z) \subset Z_E^2$$

$(z = re^{i\theta})$. Then

$$u(z) = a_0 + \sum_1^\infty r^n (a_n \cos n\theta - b_n \sin n\theta)$$

and

$$v(z) = b_0 + \sum_1^\infty r^n (b_n \cos n\theta + a_n \sin n\theta).$$

Let R be the subspace of U_E^2 generated by the orths

$$\left\{ \frac{r^n \cos n\theta}{\sqrt{\alpha_n}}, \frac{r^n \sin n\theta}{\sqrt{\alpha_n}} \right\}_1^\infty.$$

Define an isometric operator V in R as follows:

$$V(r^n \cos n\theta) = r^n \sin n\theta, \qquad V(r^n \sin n\theta) = -r^n \cos n\theta \qquad (n = 1, 2, \ldots). \tag{7}$$

Then, obviously,

$$V(u(z) - a_0) = v(z) - b_0.$$

According to (1), (2), to the operator V there corresponds the kernel

$$V(z, \zeta) = \sum_1^\infty \alpha_n^{-1} (\rho r)^n \sin n(\varphi - \theta) = -\operatorname{Im} E(z\bar{\zeta}).$$

$(z = re^{i\theta},\ \zeta = \rho e^{i\varphi})$. Taking into account that $(a_0, V(z, \zeta)) = 0$, we have

$$v(\zeta) - b_0 = (u(z), V(z, \zeta)). \tag{8}$$

Multiplying (8) by i and adding to (6), we obtain

$$u(\zeta) + iv(\zeta) - ib_0 = (u(z), \operatorname{Re} E(z\bar{\zeta}) - i \operatorname{Im} E(z\bar{\zeta})) = (u(z), E(z\bar{\zeta})).$$

Theorem 4. Every function $f(z) \subset Z_E^2$ is recovered from its real part by the formula

$$f(z) = i \operatorname{Im} f(0) + (u(\zeta), E(\bar{\zeta} z))_{U_E^2}, \tag{9}$$

where the scalar product in (9) is to be understood in the sense

$$(u, \tilde{u} + i\tilde{v}) = (u, \tilde{u}) + i(v, \tilde{v}) \quad (u, \tilde{u}, v \subset U_E^2).$$

We note that in operator form formula (9) can be written as

$$f(z) - i \operatorname{Im} f(0) = [I + iV(I - P)]u(z) \quad (u(z) = \operatorname{Re} f(z)), \tag{10}$$

where I is the identity operator; V is the isometric operator defined in (7), and P is the projecting operator $Pu(z) = u(0)$.

For simplicity let $R = 1$. Then, for

$$E(z) = \frac{1+z}{2(1-z)},$$

$Z_E^2 = H_2$ of Riesz, $U_E^2 = U_2$ of Walsh$^{(6)}$, and $E(\bar{z}\zeta)$ coincides with the kernel from Schwarz' s formula, while the scalar product is expressed by an integral over the unit circle. To obtain from this the usual Schwarz formula, it is enough to note that if $f(z)$ is analytic in C_1, then $f(kz) \subset H_2$ of Riesz for $k < 1$. The integral from formula (9) will now be taken along a circle of radius $r < 1$. Formula (9) also gives a parametric representation of the class Z_E^2 through U_E^2. If U_E^2 is embedded in a broader real Hilbert space $\tilde{U}$, then we obtain:

Theorem 5. Every function $f(z) \subset Z_E^2$ is representable in the form

$$f(z) = i \operatorname{Im} f(0) + (\tilde{u}, E(\bar{\zeta} z))_{\tilde{U}} \quad (\tilde{u} \subset \tilde{U}). \tag{11}$$

Among all functions $\tilde{u}$ representing $f(z)$ by formula (11), the function $\tilde{u} = u(z) = \operatorname{Re} f(z)$ has the least norm in $\tilde{U}$.

In conclusion we also note that, with additional isometric mappings of various spaces U_E^2, in concrete cases we could obtain integral representations of the spaces Z_E^2 by real functions subject to one or another condition of quadratic summability.

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Note: Figure translations are in progress. See original paper for figures.