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Abstract

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ON THE SOLUTION OF A PROBLEM FOR THE SCHRÖDINGER EQUATION IN INFINITE THREE-DIMENSIONAL SPACE

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One of the important problems of quantum mechanics is the scattering problem. It consists in finding, in the whole Euclidean space, a solution of the equation

$$-\Delta u + q(x)u - \lambda^2 u = 0, \tag{1}$$

satisfying certain conditions at infinity and in neighborhoods of the singular points of $q(x)$, if such points exist. The number λ^2 is assumed to be real and not coinciding with an eigenvalue of equation (1).

The question of existence and uniqueness of the solution of the scattering problem was first investigated with full rigor, for the case of a continuous potential $q(x)$, by A. Ya. Povzner ([1]).

The aim of the present work is to consider the case in which the potential $q(x)$ has singularities. Equation (1) is considered in three-dimensional Euclidean space. The proof of existence and uniqueness is carried out on the basis of an investigation of the resolvent kernel. A number of properties of the resolvent are proved, and estimates of the resolvent kernel in neighborhoods of the singular points are given.

Theorem 1. Let $q(x)$ be continuous in E, except for a finite or countable set of points having only isolated limit points (no restrictions are imposed on the behavior of $q(x)$ at infinity and in neighborhoods of the singular points). Suppose, further, that there exists a self-adjoint extension Af of the operator $Bu = -\Delta u + qu$. Then, for any regular point λ of the operator Af, its resolvent has the form

$$R_\lambda u = \int_E H(x, y; \lambda) u(y)\, dy; \tag{2}$$

$$\int_E |H(x, y; \lambda)|^2\, dy < N(Z, \lambda), \qquad x \subset Z, \tag{3}$$

where Z is any bounded domain containing no singular points.

In order to obtain more delicate properties of the resolvent kernel, above all estimates in neighborhoods of the singular points of $q(x)$, it is necessary to impose certain restrictions on $q(x)$.

Let $q(x)$, with the exception of a finite number of singular points $P_1, \dots, P_n$, be continuous, once continuously differentiable, and

$$q(x) = O\left(\frac{1}{|x|^{3.5+\varepsilon}}\right), \qquad x \to \infty, \tag{4}$$

where ε is an arbitrary positive number, while in a neighborhood of the singular points

$$q(x) \underset{s=1,2,\dots,n}{=} O\left(\frac{1}{|x-P_s|^{\alpha_s}}\right), \qquad \alpha_s < 1.5. \tag{5}$$

In what follows we shall always assume that $q(x)$ satisfies these conditions. Then the operator $Bu = -\Delta u + qu$ has zero deficiency indices [(2)] and therefore a unique self-adjoint extension Af. Below we shall consider the resolvent of this self-adjoint extension.

It can be proved that the resolvent kernel $H(x, y; \lambda^2)$, for $y \neq P_s$, satisfies the integral equation

$$f(x) + \frac{1}{4\pi} \int_E \frac{e^{-i\lambda|x-s|}}{|x-s|} q(s) f(s)\, ds = \frac{e^{i\lambda(x-y)}}{4\pi|x-y|}. \tag{6}$$

For the further study of the resolvent, some Banach-type spaces are introduced.

Let y be an arbitrary point not coinciding with P_s $(s = 1, 2, \dots, n)$, and let K_s be nonintersecting balls with centers at the points P_s, not containing the point y. Let the incomplete normed linear space $B^0_{y,K_1\dots K_n}$ consist of all functions of the form

$$f(x, y) = \frac{a}{|x-y|} + u(x),$$

where a is a constant number; $u(x)$ is continuous in E; $u(x) \to 0$ uniformly as $x \to \infty$, and

$$\|f\|_{B^0_{y,K_1\dots K_n}} = |a| + \sup_{x \subset E - \sum_s K_s} |u(x)| + \left(\sum_{s=1}^{n} \int_{K_s} u^2(x)\, dx\right)^{1/2}. \tag{7}$$

We shall denote the completion of the space $B^0_{y,K_1\dots K_n}$ by $B_{y,K_1\dots K_n}$. Let us introduce the complete space B_y, which consists of the same set of functions as $B^0_{y,K_1\dots K_n}$, but whose norm is introduced differently:

$$\|f\|_{B_y} = |a| + \sup_{x\subset E} |u(x)|. \tag{8}$$

Finally, by B we shall denote the subspace of B_y obtained from B_y by putting $a = 0$. It is proved that in these spaces, for the integral operator (6), the Fredholm alternative is valid. Using the fact that the operator under consideration has zero deficiency indices, this alternative can be formulated in terms of the spectrum of the self-adjoint extension Af. Namely, the equation

$$f(x) + \frac{1}{4\pi}\int_E \frac{e^{i\lambda(x-s)}}{|x-s|}\, q(s)f(s)\, ds = g(x) \tag{9}$$

for any $g(x)$ from the spaces under consideration has a unique solution (belonging to the corresponding spaces) if and only if λ^2 is not an eigenvalue.

Since the resolvent kernel satisfies equation (6) and $H(x, y; \lambda^2) \subset B_{y,K_1\dots K_n}$, it follows from the uniqueness of the solution of equation (6) and from the fact that

$$B \subset B_y \subset B_{y,K_1\dots K_n}, \tag{10}$$

it follows that

$$H(x, y; \lambda^2) \subset B_y; \tag{11}$$

$$h(x, y; \lambda^2) \subset B, \tag{12}$$

where

$$h(x, y; \lambda^2) = H(x, y; \lambda^2) - \frac{1}{4\pi(x-y)}.$$

From what has been said above it follows that the integral operator (9), for λ^2 not an eigenvalue of Af, has bounded inverses in the spaces under consideration. On the basis of this fact, estimates of the resolvent kernel in a neighborhood of the singular points are obtained. Thus, for example, the following theorem is valid.

Theorem 2.

$$\left|h(x,y;\lambda^2)\right|_{s=1,2,\dots,n} < \begin{cases} M, & \alpha_s < 1; \\ \dfrac{M}{|y-P_s|^{\alpha_s-1}}, & \alpha_s > 1; \\ M|\ln|y-P_s||, & \alpha_s = 1, \end{cases} \tag{13}$$

where $x \subset E$; M is a constant; α_s are the exponents in equality (5).

Analogously, the corresponding estimates are obtained for the derivatives of the resolvent kernel.

Moreover, it is proved that if λ^2, while regular, tends to a point λ_0 of the purely continuous spectrum of the operator Af, then $H(x,y;\lambda^2)$ tends uniformly, for x and y belonging to any bounded domain not containing the singular points $q(x)$, to some function $\widetilde{H}(x,y;\lambda^2)$.

Introduce the set of functions $K_{\alpha_1\dots\alpha_n}$ ($\alpha_1,\dots,\alpha_n$ are the exponents in equality (5)), which consists of all functions satisfying the conditions:

1) $v(x)$ is twice continuously differentiable in $E - \sum_{s=1}^{n} P_s$;
2) $v(x)$ satisfies the radiation conditions;
3)

$$v(x) = O_{s=1,2,\dots,n}\left(\frac{1}{|x-P_s|^{3-\alpha_s-\varepsilon}}\right), \qquad \frac{\partial v}{\partial x} = O_{s=1,2,\dots,n}\left(\frac{1}{|x-P_s|^{2-\varepsilon}}\right), \tag{14}$$

where ε is an arbitrary positive number.

We shall seek the solution of equation (1) in the form

$$u(x) = e^{i\lambda(\omega,x)} + v(x),$$

where $v(x) \subset K_{\alpha_1\dots\alpha_n}$ and ω is a given unit vector. Then equation (1) is reduced to the form:

$$-\Delta u + q(x)v - \lambda^2 v = -q(x)e^{i\lambda(\omega,x)}. \tag{15}$$

On the basis of the properties of the resolvent one can prove the following theorem:

Theorem 3. If $q(x)$ satisfies conditions (4) and (5), then equation (15), for λ^2 not coinciding with an eigenvalue of the operator Af, has in the class $K_{\alpha_1\dots\alpha_n}$ a unique solution, and this solution has the form

$$v(x) = -\int_E \widetilde{H}(x, y; \lambda^2) e^{i\lambda(\omega, y)} q(y)\, dy, \tag{16}$$

where $\widetilde{H}(x, y; \lambda^2)$, for regular λ^2, is the resolvent kernel, and for λ^2 belonging to the continuous spectrum, coincides with the limiting kernel mentioned above.

Moreover,

$$v(x) = O(1), \qquad \underset{\substack{x \to P_s \\ s=1,2,\ldots,n}}{\frac{\partial v}{\partial x}} = \begin{cases} O(1), & \alpha_s < 1, \\ O\left(\frac{1}{|xP_s|^{\alpha_s - 1}}\right), & \alpha_s > 1; \\ O(\ln|x - P_s|), & \alpha_s = 1. \end{cases} \tag{17}$$

The behavior of the solution and of the resolvent kernel at infinity (radiation conditions) can be considered analogously to the way this was done in paper ([2]).

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CITED LITERATURE

[1] A. Ya. Povzner, *Matem. sborn.*, **32** (74), 1 (1953).
[2] K. Friedrichs, *Math. Ann.*, **109**, No. 4-5 (1934).

Note: Figure translations are in progress. See original paper for figures.