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Abstract

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MATHEMATICS

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ON UNIQUENESS IN THE INVERSE PROBLEM OF POTENTIAL THEORY

(Presented by Academician M. A. Lavrent'ev, 15 I 1957)

1. P. S. Novikov ⁽¹⁾ proved the uniqueness of the solution of the inverse problem of potential in the class of star-shaped bodies of constant density. Further results on this question were obtained in the works ⁽²⁻⁴⁾. In the present note new sufficient conditions are given for uniqueness of the solution of the inverse problem of logarithmic potential.

Theorem 1. *If the boundaries C_1 and C_2 of two distinct convex domains D_1 and D_2 lie in the circular annulus*

$$R^2 \leq x^2 + y^2 \leq 2R^2, \quad (1)$$

where R is an arbitrary positive number, then the exterior logarithmic potentials of these domains, when they are filled with masses of arbitrary positive density $\mu(x, y)$, cannot coincide identically.

Theorem 2. *Let the boundaries of two distinct domains, star-shaped with respect to the pole 0, have in the polar coordinate system (r, φ) the equations $r = r_i(\varphi)$ ($i = 1, 2$), and let*

$$|\log r_i(\varphi_1) - \log r_i(\varphi_2)| \leq K|\varphi_1 - \varphi_2|, \quad (2)$$

where $K = \tan(\pi/8) \simeq 0.4142$. Then the exterior potentials of these domains, for arbitrary positive density, cannot coincide identically.

Theorem 3. *Let a simply connected domain E , bounded by a Jordan curve, be given in the plane $z = x + iy$, and let the function $\zeta = f(z)$ map the domain E conformally and one-to-one onto the disk $|\zeta| < 1$. Assign to each point $z \in E$, as its curvilinear coordinates, the numbers $r = |f(z)|$ and $\varphi = \arg f(z)$. If the boundaries of two distinct domains lying in E have, in the indicated system of curvilinear coordinates, the equations $r = r_1(\varphi)$ and $r = r_2(\varphi)$ ($0 \leq \varphi \leq 2\pi$), and*

the functions $r_i(\varphi)$ ($i = 1, 2$) satisfy condition (2), then the exterior potentials of these two domains, for arbitrary density $\mu > 0$, are distinct.

Thus, Theorem 3 gives conditions for uniqueness of the solution of the inverse problem of potential in classes of domains different from the class of star-shaped ones. Of particular importance here is the case when the density of the mass distribution is constant.

2. Let us prove Theorem 1. We assume that neither of the domains D_1 and D_2 contains the other entirely within itself. Otherwise the theorem is trivial. For the proof it is sufficient ⁽¹⁾ to construct in the domain $D = D_1 \cup D_2$ such a bounded harmonic function $u(x, y)$ that

$$\iint_{D_1} u(x, y) \mu(x, y) dx dy - \iint_{D_2} u(x, y) \mu(x, y) dx dy \neq 0. \quad (3)$$

We shall define the required function $u(x, y)$ as the solution of a Dirichlet problem in another, generally speaking smaller, domain G , and then extend it to the whole domain D .

Let C be the boundary of the domain D , lying in the circular annulus (1); let A be a component, i.e. a maximal connected open part of the set $D_1 \setminus D_2$ (or $D_2 \setminus D_1$); let c be the component adjacent to A of the set $C \cap (C_1 \setminus C_2)$ (or $C \cap (C_2 \setminus C_1)$); let Δ be the straight-line segment joining the endpoints P_1 and P_2 of the arc c . Suppose the coordinate axes are chosen so that Δ lies on the line $y = y_0$, and that above Δ , i.e. for $y > y_0$, there are no points of intersection of C_1 and C_2 .

First consider the case where $y_0 \geq 0$. Through the endpoints $P_1(x_1, y_0)$ and $P_2(x_2, y_0)$ of the segment Δ , draw the lines t_1 and t_2 tangent to the circle $x^2 + y^2 = R^2$ below Δ . Let B be the domain of the half-plane $y \geq y_0$, bounded by the segment Δ , the lines t_1 and t_2 , and, if t_1 and t_2 do not intersect inside the circle $x^2 + y^2 < 2R^2$, by another of its circles. In view of the convexity of the domains D_1 and D_2 , the domain A is wholly situated in B and may, in particular, coincide with it. Let, further, α and β be arcs of circles joining the points P_1 and P_2 , lying respectively above and below Δ , and such that α completely encompasses the domain B , forming with Δ the smallest possible angle φ ; β touches the line t_i at that one of the points P_i ($i = 1, 2$) which has the larger absolute value of the abscissa x_i . We denote by ψ the angle between Δ and β . If, in particular, $|x_1| \leq R$ and $|x_2| \leq R$, then as β we take the semicircle, i.e. $\psi = \pi/2$.

Observe that, in view of condition (1), the inequality

$$\varphi \leq 3\psi \quad (4)$$

is always satisfied.

Finally, let γ be the arc of a circle joining P_1 and P_2 and bisecting the angle between α and β . We shall say that the arcs c and γ correspond to one another. In the case where $y_0 < 0$, we shall regard the upper semicircle γ , constructed on Δ as on a diameter, as corresponding to the arc c .

We now take the boundary C of the domain D , and, replacing each component of the sets $C \cap (C_1 \setminus C_2)$ and $C \cap (C_2 \setminus C_1)$ by the corresponding arc of a circle γ , obtain a closed curve Γ . In the domain G , bounded by the curve Γ , define the harmonic function $u(x, y)$ as the solution of the Dirichlet problem with the following boundary data: $u(x, y)$ is equal to 1 on the common part of C and Γ , and also on those arcs γ for which the corresponding arcs $c \subset C_1$, and is equal to -1 on the arcs γ for which $c \subset C_2$. Under the additional condition of boundedness, $u(x, y)$ is determined uniquely by these data.

We shall show that the function $u(x, y)$ can be extended to the whole domain D in such a way that it will be positive everywhere in $D_1 \setminus D_2$ and negative in $D_2 \setminus D_1$, and, consequently, will satisfy condition (3) for any positive density $\mu(x, y)$. Consider an arc of a circle γ on which $u(x, y) = 1$, and suppose that we have the first case, i.e. $y_0 \geq 0$. Then the circular quadrilateral bounded by the arcs β and γ (or, more briefly, the quadrilateral $\beta\gamma$) will lie in the domain G and, after inversion in the arc γ , will pass into the quadrilateral $\alpha\gamma$. This means that $u(x, y)$ can be extended as a regular harmonic function to the quadrilateral $\alpha\gamma$, and all its continued values will be greater than one. If, in addition, γ lies below Δ , then in the quadrilateral $\alpha\Delta$ we shall have $u(x, y) > 1 > 0$. If γ lies above Δ , consider in the quadrilateral $\beta\gamma$ the bounded harmonic function $v(x, y)$, equal to 1 on γ and -1 on β . It is constant on each arc of a circle joining the endpoints of Δ , and is equal to zero on the one of them which bisects the angle between β and γ . This last arc, in view of inequality (4), cannot lie above Δ , i.e. $v(x, y) > 0$ in the quadrilateral $\gamma\Delta$. But $\beta\gamma \subseteq G$. Consequently, $u(x, y) \geq -1$ on β and $u(x, y) \geq v(x, y)$

in $\beta\gamma$, i.e., $u(x, y) > 0$ throughout the entire biangle $\alpha\Delta$. In view of the inclusion $A \subseteq B \subset \alpha\Delta$, our assertion is proved for $y_0 \geq 0$.

If $y_0 < 0$, then $u(x, y)$ can be extended to the whole half-plane $y \geq y_0$, with $u(x, y) > 0$ everywhere in this half-plane except, possibly, for the biangle $\delta\Delta$, where δ is the arc of a circle forming an angle $\pi/4$ with Δ . It remains to observe that the corresponding domain A lies outside the biangle $\delta\Delta$. Thus, $u(x, y) > 0$ everywhere in $D_1 \setminus D_2$. It is proved similarly that $u(x, y) < 0$ everywhere in $D_2 \setminus D_1$. The theorem is proved.

We note that Theorem 1 remains valid if, while preserving the common parts of the boundaries C_1 and C_2 of the domains D_1 and D_2 , we replace their parts lying in the biangle $\alpha\Delta$ by arbitrary Jordan arcs, also situated in $\alpha\Delta$, joining its vertices P_1 and P_2 and not intersecting inside $\alpha\Delta$. Similar remarks can be made for Theorems 2 and 3.

3. In the classes of domains considered in Theorems 1-3, regarded as compact classes, the solution of the inverse problem of potential theory ⁽⁵⁾ is stable

in the following form. Suppose that the values of the potential $V(x, y)$ are given on the x -axis, and that the perturbing domain D is contained in some prescribed bounded domain E of the half-plane $y < -1$. Consider the collection of domains $\{D\}$ lying in E and satisfying the conditions of one of Theorems 1-3. Define the degree of closeness of domains by the number

$$\max_{\varphi \in [0, 2\pi]} |r_1(\varphi) - r_2(\varphi)|$$

and of potentials by the number

$$\sup_{-\infty < x < \infty} \frac{|V_1(x, 0) - V_2(x, 0)|}{\log \sqrt{(x - x_0)^2 + y_0^2}},$$

where (x_0, y_0) is a fixed point of the domain E .

Theorem 4. *For any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that, if the values of the potentials $V_1(x, y)$ and $V_2(x, y)$ of two domains of the class $\{D\}$ differ for $y = 0$ by less than $\delta(\varepsilon)$, then the domains themselves are separated from one another by less than ε .*

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CITED LITERATURE

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