ON THE QUESTION OF THE TYPE OF EQUATIONS OF THE THEORY OF PLASTICITY

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Abstract

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THEORY OF ELASTICITY

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ON THE QUESTION OF THE TYPE OF EQUATIONS OF THE THEORY OF PLASTICITY

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V. V. Sokolovskii ([1]) showed that, in the case of a plane stress state, when the plasticity condition is used in the Mises form, plastic equilibrium can be described by both hyperbolic and elliptic equations. In this note the general connection is clarified between the form of the plasticity condition and the type of the corresponding equations in plane strain and plane stress states.

As is known, in the plane case the plasticity condition can always be written in the form

$$S^2 = f(p),$$

where

$$S^2 = \frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2;$$

$$p = \frac{1}{2}(\sigma_x + \sigma_y)$$

are two independent invariants of the stress tensor; f is a function determining the form of the plasticity condition. Geometrically, this condition represents a family of limiting Mohr circles in the σ, τ plane.

We wish to show that the equations of the theory of plasticity will be of hyperbolic type for those values of the parameter p for which this family of circles has an envelope, parabolic when the envelope degenerates into a point, and elliptic for the remaining values of p (Fig. 1).

For this purpose let us consider the system of equations of plasticity and determine under what condition it can have real characteristics. To this end, in the equilibrium equations

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=0,\qquad \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}=0$$

we shall regard τ_{xy} as a known function of σ_x and σ_y, determined by the plasticity condition (1). Then the equilibrium equations will constitute a system of two equations for the two unknowns σ_x and σ_y of the form

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial\sigma_x}\frac{\partial\sigma_x}{\partial y}+\frac{\partial\tau_{xy}}{\partial\sigma_y}\frac{\partial\sigma_y}{\partial y}=0,$$

$$\frac{\partial\tau_{xy}}{\partial\sigma_x}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial\sigma_y}\frac{\partial\sigma_y}{\partial x}+\frac{\partial\sigma_y}{\partial y}=0.$$

Adjoining here the relations

$$d\sigma_x=\frac{\partial\sigma_x}{\partial x}\,dx+\frac{\partial\sigma_x}{\partial y}\,dy,$$

$$d\sigma_y=\frac{\partial\sigma_y}{\partial x}\,dx+\frac{\partial\sigma_y}{\partial y}\,dy.$$

we obtain a system of four equations for the derivatives of σ_x and σ_y with respect to x and y. Equating the determinant of this system to zero, we obtain the differential equations of the characteristic projections

$$\frac{dy}{dx}=-\frac{1}{2\frac{\partial\tau_{xy}}{\partial\sigma_y}}\left(1\pm\sqrt{1-4\frac{\partial\tau_{xy}}{\partial\sigma_x}\frac{\partial\tau_{xy}}{\partial\sigma_y}}\right).$$

In order that the system be hyperbolic or parabolic, it is necessary and sufficient that the inequality

$$\frac{\partial\tau_{xy}}{\partial\sigma_x}\frac{\partial\tau_{xy}}{\partial\sigma_y}\le\frac{1}{4}$$

be satisfied.

Let us see what restrictions this inequality imposes on the function $f(p)$ in condition (1). Using (1), (2), and (3), we can write

$$\tau_{xy}=\sqrt{f\left[\frac{1}{2}(\sigma_x+\sigma_y)\right]-\frac{1}{4}(\sigma_x-\sigma_y)^2}.$$

Fig. 1. Relative arrangement of Mohr circles and the envelope. Circles 1, 2, 3 have an envelope; for circles 3, 4, 5 it degenerates into a point; circles 5, 6 have no envelope

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Differentiating this expression with respect to σ_x and σ_y and substituting into (5), after elementary operations, we obtain that inequality (5) is equivalent to the inequality

$$\left[\frac{df(p)}{dp}\right]^2 \leq 4f(p).$$

Fig. 1. Relative arrangement of Mohr circles and the envelope. Circles 1, 2, 3 have an envelope; for circles 3, 4, 5 it degenerates into a point; circles 5, 6 have no envelope.

We shall now show that inequality (6) is at the same time the condition for the existence of an envelope. Recalling that S^2 is the square of the radius of the limiting circle of plasticity, and $(p, 0)$ are the coordinates of its center, we can rewrite (1) in the coordinates σ, τ in the form

$$(\sigma - p)^2 + \tau^2 = f(p).$$

Differentiating this equality with respect to p and equating the derivative to zero, we obtain

$$\sigma = p - \frac{1}{2}f'(p).$$

To obtain a parametric representation of the envelope, substitute this expression into (7). We find

$$\tau = \sqrt{f(p) - \frac{1}{4}\left[\frac{df(p)}{dp}\right]^2},$$

which is possible only if (6) is satisfied. Thus, the condition for the existence of an envelope coincides with the condition for hyperbolicity of system (4). If equality holds in (6) on some interval of variation of p, then system (4) is parabolic, and the envelope degenerates into a point. This proves our assertion.

As an example, consider the Mises plasticity condition mentioned above. In the case of a plane stress state it is written in the form

$$S^2 = \frac{\sigma_s^2 - p^2}{3} = f(p),$$

where σ_s is the yield limit.

Let us use inequality (6). We have

$$f'(p) = -\frac{2}{3}\,p;$$

$$\frac{4}{9}\,p^2 < \frac{4}{3}\,(\sigma_s^2 - p^2),$$

whence

$$|p| < \frac{\sigma_s\sqrt{3}}{2}.$$

For values of p satisfying this inequality, plastic equilibrium is described by hyperbolic equations. For

$$\frac{\sigma_s\sqrt{3}}{2} < |p| < \sigma_s$$

the equations pass over into the elliptic type.

An example of the parabolic type is furnished by a plane stress state under the Saint-Venant plasticity condition in the region $\sigma_1\sigma_2 > 0$.

Indeed, in this case the plasticity condition is written in the form

$$S^2 = (\sigma_s - p)^2 = f(p).$$

Then

$$f'(p) = -2(\sigma_s - p), \qquad \left[\frac{df(p)}{dp}\right]^2 = 4f(p).$$

System (4) under this condition has one family of real characteristics.

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CITED LITERATURE

1. V. V. Sokolovskii, *Theory of Plasticity*, 1950.

Note: Figure translations are in progress. See original paper for figures.