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Abstract

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ON SOME NONLINEAR OPERATORS IN ORLICZ SPACES

(Presented by Academician P. S. Aleksandrov on 30 V 1957)

Let $M(u)$ and $N(v)$ be two mutually complementary Young N'-functions, and let G be some compact set in a finite-dimensional space. By L_M^* we denote the Orlicz space $L_M^*(G)$, i.e. the linear hull of the set L_M of functions $u(x)$ $(x \in G)$ for which

$$\rho(u; M) = \int_G M[u(x)]\, dx.$$

The norm in L_M^* is defined by the equality

$$\|u\|_M = \sup_{\rho(v;N)\leq 1} \int_G u(x)v(x)\, dx. \tag{1}$$

By E_M we denote the subspace of L_M^* that is the closure in L_M^* of the set of functions bounded on G; by $\hat{L}_M^*$ we denote the Orlicz space $L_M^*(\hat{G})$, where $\hat{G}$ is the topological product $G \times G$. The space $\hat{E}_M$ is defined analogously.

One says that an N'-function $M(u)$ satisfies the Δ_2-condition if, for large values of u, $M(2u) \leq KM(u)$. One says that an N'-function $M(u)$ satisfies the Δ'-condition if, for large values of u and v,

$$M(uv) \leq CM(u)M(v).$$

In the present note we study some properties of operators and functionals defined on L_M^*.

1. Let $f(x, u)$ $(x \in G, \ -\infty < u < \infty)$ be a function satisfying the Carathéodory conditions, i.e. continuous in u for each x and measurable in x for each fixed u. By $\mathbf{f}$ we denote the operator defined by the equality

$$\mathbf{f}u(x) = f[x, u(x)].$$

In note (^{1}) criteria were established for differentiability of the operator $\mathbf{f}$ at a certain isolated point of an Orlicz space. Here we indicate one of the conditions for differentiability of the operator $\mathbf{f}$ in some ball.

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Theorem 1. Let the N'-functions $M_1(u)$, $M_2(u)$, and $\Phi(u)$ satisfy the following condition: there exist N'-functions $R(u)$ and $Q(u)$, complementary to one another, such that for large values of the argument u and for some $\alpha > 0$

$$R(\alpha u) < M_2^{-1}[M_1(u)], \qquad Q(\alpha u) < M_2^{-1}[\Phi(u)], \tag{2}$$

or, if the N'-function $M_2(u)$ satisfies the Δ'-condition,

$$R(\alpha u) < M_1[M_2^{-1}(u)], \qquad Q(\alpha u) < \Phi[M_2^{-1}(u)]. \tag{3}$$

Here $M_2^{-1}(v)$ is the function inverse to $M_2(u)$. Let the function $f(x, u)$, together with its derivative $f'_u(x, u)$, satisfy the Carathéodory conditions; moreover, let the operator $\mathbf{f}$ act from some ball $T \subset L^*_{M_1}$ into the space $L^*_{M_2}$, and let the operator

$$\mathbf{f}_1 u(x) = f'_u[x, u(x)]$$

act from the ball T into the space L^*_Φ and be continuous.

Then the operator f is Fréchet differentiable at every interior point of the ball T, and its Fréchet differential Bh at the point $u(x) \in T$ is determined by the equality

$$Bh(x) = f'_u[x, u(x)] \cdot h(x) \qquad (h(x) \in L^*_{M_1}).$$

In the proof of this theorem the following assertion was used, which, it seems to us, is of independent interest for the general theory of Orlicz spaces.

Theorem 2. Let the N'-functions $M_1(u)$, $M_2(u)$, and $\Phi(u)$ satisfy one of the conditions (2) or (3). Then, for any pair of functions $a(x) \in L^*_\Phi$, $u(x) \in L^*_{M_1}$, the product $a(x)u(x)$ belongs to $L^*_{M_2}$, and

$$\|a(x)u(x)\|_{M_2} \leq K \|u\|_{M_1} \|a\|_\Phi,$$

where the constant K depends on the functions $a(x)$, $u(x)$.

This theorem generalizes the known assertion that under the condition

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} < 1$$

the product $a(x)u(x)$ belongs to the space L^{α_3}, where

$$a(x) \in L^{\alpha_1}, \quad u(x) \in L^{\alpha_2}, \quad \alpha_3 = \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}.$$

2. In the same note [1], conditions for the complete continuity of the nonlinear integral operator

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$$H\varphi(x) = \int_G K(x, y) f[y, \varphi(y)]\, dy$$

were established.

Consider an operator of a more general form

$$K\varphi(x) = \int_G K[x, y, \varphi(y)]\, dy. \tag{4}$$

Theorem 3. Let $M(u)$ and $N(v)$ be N'-functions complementary to each other, the second of which satisfies the Δ'-condition. Let

$$|K(x, y, u)| \le \bar{K}(x, y)[a(x) + R(|u|)] \qquad (x, y \in G,\ -\infty < u < \infty),$$

where $\bar{K}(x, y) \in \hat{E}_M$; $a(x) \in L_N$; $R(u)$ is a nonnegative function nondecreasing for $u > 0$. Finally, suppose that there exist positive numbers β, γ, and k such that, for large values of the argument,

$$N[\beta R(\gamma u)] \le kM(u). \tag{5}$$

Then the operator (4) maps some ball T of the space L^*_Φ into the space L_Φ and is completely continuous. Here $\Phi(u)$ is an arbitrary N'-function satisfying, for large values of u, the inequality

$$N[\beta R(\gamma u)] \le k\Phi(u) \le kM(u). \tag{6}$$

Recall that $K(x, y) \in \hat{E}_M$ if for all α

$$\iint_G M[\alpha K(x, y)]\, dx\, dy < \infty.$$

Under the conditions of this theorem, the operator (4) is defined on the whole space L^*_Φ if in inequalities (5) and (6) one can take γ arbitrarily large; for example, if the N'-function $\Phi(u)$ satisfies the Δ_2-condition, or if the function $R(u)$ satisfies an analogous inequality.

Conditions (5) and (6) can in some cases be written in a simpler form. For example, if the N'-function $\Phi(u)$, for large values of u, satisfies the inequality

$$N[\Phi(u)] \le \Phi(\alpha u), \tag{7}$$

then conditions (5) and (6) are fulfilled if, for large values of u,

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$$R(\alpha\gamma u) \leq k\Phi(u) \leq kM(u).$$

As $\Phi(u)$ one may also choose the function $M(u)$. In this case condition (7), generally speaking, is satisfied if $M(u)$ grows faster than any power. More precisely, condition (7) is satisfied if $M(u)$ satisfies the Δ_3-condition: for large values of u,

$$uM(u) < M(k_1 u).$$

Theorem 3 makes it possible to establish complete continuity of operators (4) containing essentially non-power nonlinearities.

Let, for example,

$$|K(x, y, u)| \leq K(x, y)e^{\alpha u} \qquad (x, y \in G, \ -\infty < u < \infty),$$

where

$$K(x, y) \leq a|\ln r|^{1-\beta_0} + b,$$

where $a, b > 0$; r is the distance between the points $x, y \in G$; $0 < \beta_0 < 1$.

Then the operator (4) is completely continuous in a certain ball of the space $L^*_{\Phi_0}$, where $\Phi_0(u) = e^{|u|} - |u| - 1$, and is completely continuous in the whole space L^*_Φ, $\Phi(u)$ being any function of the form $e^{|u|^{1+\beta}} - 1$, $0 < \beta < \beta_0$.

Under the hypotheses of Theorem 3 the function $M(u)$ grows faster than some power, since the function $N(v)$ complementary to it satisfies the Δ'-condition. We now consider the case when the N-function $M(u)$ grows more slowly than any power $|u|^{1+\alpha}$ $(\alpha > 0)$. In this case, generally speaking, the complementary function $N(v)$ satisfies the Δ_3-condition.

Theorem 4. *Let $M(u)$ and $N(v)$ be mutually complementary N-functions, the second of which satisfies the Δ_3-condition. Let*

$$|K(x, y, u)| \leq K(x, y)[a(x) + R(|u|)] \qquad (x, y \in G, \ -\infty < u < \infty),$$

where $K(x, y) \in \hat{L}_M$; $a(x) \in L^*_N$; $R(u)$ is a nonnegative function, nondecreasing for $u > 0$. Finally, suppose that there is a $C > 0$ such that, for large values of the argument, the inequality

$$R(u) < C\,\frac{M(u)}{u}.$$

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holds. Then there exists an Orlicz space L^*_Φ in which the operator (4) acts and is completely continuous.

The hypotheses of the theorem are, for example, satisfied if

$$|K(x, y, u)| \leq K(x, y)[a + \ln(|u| + 1)],$$

$$\iint_{\hat{G}} K(x, y) \ln[K(x, y) + 1] \, dx \, dy < \infty.$$

The differentiability conditions for the operator (4) in a certain ball of the space L^*_Φ can be obtained in a form analogous to Theorem 1, and finer differentiability conditions at a single point—in a form analogous to the corresponding theorem for the operators F and H in (1).

Theorems 3 and 4 make it possible to apply the general methods of nonlinear functional analysis to the investigation of equations with the operator (4), containing nonlinearities of various kinds (for example, exponential ones). Standard arguments lead to theorems on the existence of solutions, eigenfunctions, bifurcation points, etc.

3. Some general theorems on nonlinear operator equations pertain to spaces with differentiable norm. In this connection the authors considered the question of differentiability of the norm in Orlicz spaces. We note that the assertions given below admit a generalization to the so-called modulared spaces, the theory of which is being developed by a group of Japanese mathematicians headed by Nakano (see, for example, (^{2})).

In this section it is assumed that the N-function $M(u)$ has a continuously differentiable derivative $p(u)$, and that the complementary N-function $N(v)$ satisfies the Δ_2-condition.

Theorem 5*. If the derivative $p'(u)$ is monotone and positive for $u \neq 0$, then the norm (1) is a differentiable functional on E_M. The gradient Γ of the norm (1) is determined by the equality

$$\Gamma u(x) = p(k^* u(x)) \quad (u(x) \in E_M),$$

where k^* is determined by the equation

$$\int_G N[p(k^* |u(x)|)] \, dx = 1.$$

Recall that an operator Γ, acting from E_M into L^*_N, is called the gradient of the functional $F(u)$, defined on E_M, if

$$\lim_{\substack{h(x)\in E_M\\ \|h\|_M\to 0}} \frac{|F(u+h)-F(u)-(\Gamma u, h)|}{\|h\|_M} = 0.$$

As usual, (w, h) denotes the scalar product

$$(w, h) = \int_G w(x)h(x)\,dx \quad (w(x) \in L_N^*,\ \ h(x) \in E_M).$$

We note that from this theorem there follows the well-known theorem of Mazur on the formula for the gradient of the norm in L^p.

4. In an Orlicz space one can also introduce other norms, different from (1). One of such norms was widely used by Luxemburg (^{3}).

This norm $\|u\|_{(M)}$ is defined as follows:

$$\|u\|_{(M)} = \inf k, \tag{8}$$

where the infimum is taken over all such $k > 0$ for which

$$\rho\left(\frac{u}{k};\, M\right) = \int_G M\left[\frac{u(x)}{k}\right] dx \leq 1. \tag{9}$$

Theorem 6. Let the N'-function $M(u)$ have a continuous derivative $p(u)$, and let the complementary function $N(v)$ satisfy the Δ_2-condition. Then the norm defined by equalities (8) and (9) is a differentiable functional on E_M. The gradient Γ_1 of this norm is determined by the formula

$$\Gamma_1 u(x) = \frac{p\left(\dfrac{u(x)}{\|u\|_{(M)}}\right)}{\displaystyle\int_G p\left(\frac{u(x)}{\|u\|_{(M)}}\right)\frac{u(x)}{\|u\|_{(M)}}\,dx}.$$

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* In the proof of this theorem certain propositions were used that were established by the authors jointly with N. G. Shimko.

Note: Figure translations are in progress. See original paper for figures.

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