ON ANALYTIC SOLUTIONS OF THIRD-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

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ON ANALYTIC SOLUTIONS OF THIRD-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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The emergence of the analytic theory of partial differential equations is connected with the work of S. V. Kovalevskaya ([1]), where the well-known theorem on the existence and uniqueness of a solution of a system of Kovalevskaya type was proved. But Kovalevskaya' s theorem does not answer the question of the existence and uniqueness of a solution if the system or the boundary conditions are not reduced to the required form. This question was studied by Méray ([2]), N. M. Günter ([3]), S. L. Sobolev ([4,5]), and a number of other mathematicians.

Günter proposed using, for the solution of a partial differential equation, a finite-difference equation arising upon differentiating the original equation and making it possible to determine the initial values of all derivatives of the unknown function. Sobolev extended Günter' s method to a system of first-order equations.

In the present paper Günter' s method is applied to the equation

$$\Phi\left(x_1, x_2, x_3; \frac{\partial^{s_1+s_2+s_3} v}{\partial x_1^{s_1} \partial x_2^{s_2} \partial x_3^{s_3}}\right) = 0 \quad (s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0;\ s_1+s_2+s_3 \leq 3) \tag{1}$$

with boundary conditions

$$v\big|_{x_1=0} = 0, \qquad v\big|_{x_2=0} = 0, \qquad v\big|_{x_3=0} = 0. \tag{2}$$

It is required to find a solution analytic in some neighborhood of the origin. Denoting

$$\left(\frac{\partial^{l_1+l_2+l_3} v}{\partial x_1^{l_1} \partial x_2^{l_2} \partial x_3^{l_3}}\right)_{(0)} = f(l_1, l_2, l_3), \qquad \left(\frac{\partial \Phi}{\partial \dfrac{\partial^3 v}{\partial x_1^{r_1} \partial x_2^{r_2} \partial x_3^{r_3}}}\right)_{(0)} = \alpha_{r_1 r_2 r_3}$$

$$(r_1 \geq 0, \quad r_2 \geq 0, \quad r_3 \geq 0, \quad r_1 + r_2 + r_3 = 3),$$

by differentiating the original equation and the boundary conditions we obtain the difference equation

$$\sum_{\substack{r_1+r_2+r_3=3 \\ r_1,r_2,r_3\geq 0}} \alpha_{r_1 r_2 r_3} f(l_1 + r_1 - 1,\ l_2 + r_2 - 1,\ l_3 + r_3 - 1) = \psi(l_1, l_2, l_3) \qquad (3)$$

$$(l_1 > 0, \quad l_2 > 0, \quad l_3 > 0, \quad l_1 + l_2 + l_3 = n)$$

under the conditions

$$f(l_1, l_2, l_3) = 0 \quad (l_1 l_2 l_3 = 0), \qquad (4)$$

where $\psi(l_1, l_2, l_3)$ is expressed in terms of derivatives of v of order lower than $l_1 + l_2 + l_3$. Using (3) and (4), one can, knowing the derivatives of v up to order $n - 1$ inclusive, obtain a difference equation for determining the derivatives of order n, and so on. In doing this, it is necessary to find out whether the obtained difference equation has a solution and whether the formally constructed power series has a nonzero radius of convergence.

Let us introduce the notation

$$\sum_{\substack{r_1+r_2+r_3=3 \\ r_1,r_2,r_3\geq 0}} \alpha_{r_1 r_2 r_3} x_1^{r_1} x_2^{r_2} x_3^{r_3} = P(x_1, x_2, x_3).$$

The coefficients P and the variables x_1, x_2, x_3 are considered in the complex plane.

Theorem. Suppose

1) For no x_1, x_2, x_3 satisfying the condition $|x_1| = |x_2| = |x_3| = 1$ does P vanish.

2) For any fixed x_2 and x_3 satisfying the condition $|x_2| = |x_3| = 1$, there exists a unique value x_1 such that

$$P(x_1, x_2, x_3) = 0, \qquad \partial P / \partial x_1 \neq 0; \quad |x_1| < 1.$$

We shall denote this value of x_1 by $x_1(x_2, x_3)$. Analogously, introduce the notations $x_2(x_3, x_1)$ and $x_3(x_1, x_2)$.

3) The system of integral equations

$$\varphi_{12}\left(\frac{x_1}{x_2}\right) + \oint_{|x_2^{(0)}|=1} \oint_{|x_3^{(0)}|=1} \varphi_{13}\left(\frac{x_1^{(0)}}{x_3^{(0)}}\right) \frac{\frac{x_1}{x_2}\, x_2\left(x_3^{(0)}, x_1^{(0)}\right)}{x_1^{(0)} - \frac{x_1}{x_2}\, x_2\left(x_3^{(0)}, x_1^{(0)}\right)} \frac{dx_1^{(0)}\, dx_3^{(0)}}{x_1^{(0)} x_3^{(0)}} = 0,$$

$$\varphi_{13}\left(\frac{x_1}{x_3}\right) + \oint_{|x_1^{(0)}|=1} \oint_{|x_2^{(0)}|=1} \varphi_{12}\left(\frac{x_1^{(0)}}{x_2^{(0)}}\right) \frac{\frac{x_1}{x_3}\, x_3\left(x_1^{(0)}, x_2^{(0)}\right)}{x_1^{(0)} - \frac{x_1}{x_3}\, x_3\left(x_1^{(0)}, x_2^{(0)}\right)} \frac{dx_1^{(0)}\, dx_2^{(0)}}{x_1^{(0)} x_2^{(0)}} = 0 \tag{5}$$

has no nontrivial solutions. An analogous requirement is imposed on the two other systems of integral equations obtained by cyclic permutations of all indices.

Then there exists a number n_0, depending only on the coefficients of P, such that for every $n \geq n_0$ equation (3) under conditions (4) has a solution satisfying the inequality

$$|f(l_1, l_2, l_3)| \leq C \max |\psi(l_1, l_2, l_3)|, \tag{6}$$

where C is independent of n.

Proof. First we solve (3), disregarding conditions (4). Choose an arbitrary point $(l_1^{(0)}, l_2^{(0)}, l_3^{(0)})$ and require that

$$\psi(l_1, l_2, l_3) = \begin{cases} 1, & \text{if } (l_1 - l_1^{(0)})^2 + (l_2 - l_2^{(0)})^2 + (l_3 - l_3^{(0)})^2 = 0, \\ 0, & \text{if } (l_1 - l_1^{(0)})^2 + (l_2 - l_2^{(0)})^2 + (l_3 - l_3^{(0)})^2 > 0. \end{cases} \tag{7}$$

Form

$$f(l_1, l_2, l_3;\, l_1^{(0)}, l_2^{(0)}, l_3^{(0)}) = \frac{1}{(2\pi i)^3} \oint_{|x_1|=1} \oint_{|x_2|=1} \oint_{|x_3|=1} \frac{x_1^{l_1 - l_1^{(0)}}\, x_2^{l_2 - l_2^{(0)}}\, x_3^{l_3 - l_3^{(0)}}\, dx_1 dx_2 dx_3}{P(x_1, x_2, x_3)}.$$

This integral exists by condition 1) of the theorem. As a function of l_1, l_2, l_3 it satisfies (3) for $\psi(l_1, l_2, l_3)$ specified by (7). The solution of (3) for arbitrary $\psi(l_1, l_2, l_3)$ has the form

$$f_1(l_1, l_2, l_3) = \sum_{\substack{l_1^{(0)} + l_2^{(0)} + l_3^{(0)} = n \\ l_1^{(0)}, l_2^{(0)}, l_3^{(0)} \geq 0}} \psi(l_1^{(0)}, l_2^{(0)}, l_3^{(0)}) f(l_1, l_2, l_3;\, l_1^{(0)}, l_2^{(0)}, l_3^{(0)}).$$

Making in equation (3) the substitution

$$f(l_1, l_2, l_3) = f_1(l_1, l_2, l_3) + f_2(l_1, l_2, l_3), \tag{8}$$

we obtain the homogeneous equation

$$\sum_{\substack{r_1+r_2+r_3=3\\ r_1,r_2,r_3\geq 0}} \alpha_{r_1r_2r_3}\, f_2(l_1 + r_1 - 1,\ l_2 + r_2 - 1,\ l_3 + r_3 - 1) = 0 \tag{9}$$

with the nonhomogeneous boundary conditions

$$f_2(l_1, l_2, l_3) = -f_1(l_1, l_2, l_3), \quad (l_1 l_2 l_3 = 0). \tag{10}$$

To solve (9) under the conditions (10), we construct a certain system of functions. Choose an arbitrary point $(0, l_2^{(0)}, l_3^{(0)})$ and construct

$$f_1(l_1, l_2, l_3;\, l_2^{(0)}, l_3^{(0)}) = \frac{1}{(2\pi i)^2} \oint_{|x_2|=1} \oint_{|x_3|=1} x_1^{l_1}(x_2, x_3)\, x_2^{l_2 - l_2^{(0)} - 1}\, x_3^{l_3 - l_3^{(0)} - 1}\, dx_2\, dx_3.$$

Similarly we construct $f_2(l_1, l_2, l_3;\, l_3^{(0)}, l_1^{(0)})$ and $f_3(l_1, l_2, l_3;\, l_1^{(0)}, l_2^{(0)})$. In this case:

1) each of the functions f_1, f_2, f_3, as a function of l_1, l_2, l_3, satisfies (9);

2)

$$f_1(0, l_2, l_3;\, l_2^{(0)}, l_3^{(0)}) = \begin{cases} 1, & \text{if } (l_2 - l_2^{(0)})^2 + (l_3 - l_3^{(0)})^2 = 0, \\ 0, & \text{if } (l_2 - l_2^{(0)})^2 + (l_3 - l_3^{(0)})^2 > 0, \end{cases}$$

and similarly for f_2 and f_3;

3) there exists a constant k $(0 < k < 1)$ such that

$$|f_1(l_1, l_2, l_3;\, l_2^{(0)}, l_3^{(0)})| \leq k^{l_1 + |l_2 - l_2^{(0)}| + |l_3 - l_3^{(0)}|}.$$

For each point $(l_1^{(0)}, l_2^{(0)}, l_3^{(0)})$ satisfying the condition

$$l_1^{(0)} l_2^{(0)} l_3^{(0)} = 0,$$

we form the corresponding f_1, f_2, or f_3; moreover, for the point $(n, 0, 0)$ one may choose arbitrarily f_2 or f_3, and similarly for the points $(0, n, 0)$ and $(0, 0, n)$. We shall seek a solution of (9) under the conditions (10) in the form of a linear combination of the f_j. Any linear combination of them will be a solution of (9), and it remains only to satisfy the conditions (10). In doing so, complications are possible near the points $(n, 0, 0)$, $(0, n, 0)$, and $(0, 0, n)$. To avoid these complications, we subject some f_j to a transformation. Denote by p the least nonnegative integer for which

$$\frac{2k^{2p+2}}{1-k^2} = \theta < 1.$$

Consider the case $p > 0$. For $p = 0$ the transformation described need not be carried out.

Consider the set of points satisfying the condition

$$l_1^{(0)} + l_2^{(0)} + l_3^{(0)} = n, \qquad l_2^{(0)} l_3^{(0)} = 0; \qquad 0 \le l_2^{(0)} \le p, \qquad 0 \le l_3^{(0)} \le p.$$

Choose an arbitrary point from the indicated set. Form the function

$$h_1(l_1, l_2, l_3;\, l_1^{(0)}, l_2^{(0)}, l_3^{(0)}) = \sum_{m_3^{(0)}=0}^{\infty} \xi(m_3^{(0)})\, f_2(l_1, l_2, l_3;\, m_3^{(0)};\, n - m_3^{(0)}) +$$

$$+ \sum_{m_2^{(0)}=0}^{\infty} \eta(m_2^{(0)})\, f_3(l_1, l_2, l_3;\, m_2^{(0)};\, n - m_2^{(0)})$$

and require that

$$h_1(l_1, l_2, l_3;\, l_1^{(0)}, l_2^{(0)}, l_3^{(0)}) = \begin{cases} 1, & \text{if } (l_1 - l_1^{(0)})^2 + (l_2 - l_2^{(0)})^2 + (l_3 - l_3^{(0)})^2 = 0, \\ 0, & \text{if } (l_1 - l_1^{(0)})^2 + (l_2 - l_2^{(0)})^2 + (l_3 - l_3^{(0)})^2 > 0, \quad l_2^{(0)} l_3^{(0)} = 0. \end{cases}$$

This requirement represents an infinite system of linear equations, whose solution, as can be verified, exists if (5) has no nontrivial solutions. In an analogous manner we construct the functions
$h_2(l_1, l_2, l_3;\, l_1^{(0)}, l_2^{(0)}, l_3^{(0)})$ and $h_3(l_1, l_2, l_3;\, l_1^{(0)}, l_2^{(0)}, l_3^{(0)})$.

Next we seek the solution of (9) under the conditions (10) in the form

$$f_2(l_1, l_2, l_3) = \sum_{\substack{l_2^{(0)} l_3^{(0)}=0;\\ 0\le l_2^{(0)},\, l_3^{(0)}\le p}} \xi(l_1^{(0)}, l_2^{(0)}, l_3^{(0)}) h_1(l_1, l_2, l_3;\, l_1^{(0)}, l_2^{(0)}, l_3^{(0)}) +$$
$$+ \sum_{\substack{l_3^{(0)} l_1^{(0)}=0;\\ 0\le l_3^{(0)},\, l_1^{(0)}\le p}} \xi(l_1^{(0)}, l_2^{(0)}, l_3^{(0)}) h_2(l_1, l_2, l_3;\, l_1^{(0)}, l_2^{(0)}, l_3^{(0)}) +$$
$$+ \sum_{\substack{l_1^{(0)} l_2^{(0)}=0;\\ 0\le l_1^{(0)},\, l_2^{(0)}\le p}} \xi(l_1^{(0)}, l_2^{(0)}, l_3^{(0)}) h_3(l_1, l_2, l_3;\, l_1^{(0)}, l_2^{(0)}, l_3^{(0)}) + \quad (11)$$
$$+ \sum_{l_1^{(0)}=0;\, l_2^{(0)}, l_3^{(0)}>p} \xi(l_1^{(0)}, l_2^{(0)}, l_3^{(0)}) f_1(l_1, l_2, l_3;\, l_2^{(0)}, l_3^{(0)}) +$$
$$+ \sum_{l_2^{(0)}=0;\, l_3^{(0)}, l_1^{(0)}>p} \xi(l_1^{(0)}, l_2^{(0)}, l_3^{(0)}) f_2(l_1, l_2, l_3;\, l_3^{(0)}, l_1^{(0)}) +$$
$$+ \sum_{l_3^{(0)}=0;\, l_1^{(0)}, l_2^{(0)}>p} \xi(l_1^{(0)}, l_2^{(0)}, l_3^{(0)}) f_3(l_1, l_2, l_3;\, l_1^{(0)}, l_2^{(0)}).$$

Substituting the boundary conditions (10) into (11), we obtain a system of $3n$ linear algebraic equations with $3n$ unknowns (according to the number of boundary points). This system can be solved by the method of iterations. Having obtained $\xi(l_1^{(0)}, l_2^{(0)}, l_3^{(0)})$ for all boundary points, from (11) we determine $f_2(l_1, l_2, l_3)$, and then from (8) we determine $f(l_1, l_2, l_3)$. The obtained $f(l_1, l_2, l_3)$ will satisfy the estimate (6).

From the existence theorem proved above follows the uniqueness of the solution of (9) under the conditions (10), according to the general theory of linear algebraic equations.

On the basis of the estimate (6) and the uniqueness of the solution, by a method analogous to that used in ([3,4]), the existence and uniqueness of the solution of the differential equation is proved.

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Note: Figure translations are in progress. See original paper for figures.