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Abstract

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B. I. KORENBLUM

ON A NORMED RING OF FUNCTIONS WITH CONVOLUTION

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In the present note we continue the study of the ideals of the ring $L(-\infty,\infty;\alpha)$, begun by us in (^{1}). A complete description is given of the ideals of this ring with a finite hull (see Definition 3), situated inside the strip $|\operatorname{Im} z| \leqslant \alpha$. This makes it possible to solve the problem of harmonic synthesis for certain general classes of rapidly increasing functions (^{1}). The results obtained find application to homogeneous integral equations of convolution type.

1°. In accordance with the notation of note (^{1}), $L(-\infty,\infty;\alpha)$ is a commutative normed ring of measurable functions

$$f(x) \quad (-\infty < x < \infty)$$

with norm

$$\|f\| = \int_{-\infty}^{\infty} |f(x)| e^{\alpha|x|}\, dx < \infty \quad (\alpha > 0 \text{ fixed})$$

and with the convolution operation as multiplication of elements; $M(-\infty,\infty;\alpha)$ is the space of measurable functions $g(x)$ $(-\infty < x < \infty)$ with norm

$$\|g\| = \operatorname*{vrai\,max}_{-\infty<x<\infty} \{e^{-\alpha|x|}|g(x)|\} < \infty.$$

Obviously, the space $M(-\infty,\infty;\alpha)$ is conjugate to $L(-\infty,\infty;\alpha)$, and the general form of a linear functional in $L(-\infty,\infty;\alpha)$ is given by the formula

$$g(f) = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

Let us formally adjoin to the ring $L(-\infty,\infty;\alpha)$ a unit element e. The ring L_e thus obtained consists of elements $\lambda e + f(x)$, where $f(x) \in L(-\infty,\infty;\alpha)$, and λ is an arbitrary complex number; the operations of addition, multiplication, and the norm are defined in the natural way. Rings of this type were first considered by I. M. Gelfand, D. A. Raikov, and G. E. Shilov (^{2}). In particular, they gave a complete description of the maximal ideals of the ring L_e. The space of these ideals is homeomorphic to the complex strip $|\operatorname{Im} z| \leqslant \alpha$ with the point at infinity adjoined, to which corresponds the ideal $I_\infty = L(-\infty, \infty; \alpha)$. In the classical case $\alpha = 0$, the strip degenerates into the line $\operatorname{Im} z = 0$. The well-known theorem of Wiener (^{3}) asserts in this case that there does not exist a single primary ideal* contained in the ideal I_∞. Subsequently this theorem was extended by G. E. Shilov (^{4}) to more general rings of functions with convolution.

A fundamentally different situation arises in the case considered by us. From the results of note (^{1}) it follows that there exist primary ideals contained in $I_\infty = L(-\infty, \infty; \alpha)$ $(\alpha > 0)$. Below a complete description of these primary ideals is given. For convenience we introduce the following definitions.

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* An ideal (closed) is called **primary** if it is contained in only one maximal ideal.

Definition 1. Let $f(x) \in L(-\infty, \infty; \alpha)$ and

$$F(z) = F(x+iy) = \int_{-\infty}^{\infty} f(t) e^{-izt}\, dt \qquad (|y| \leqslant \alpha). \tag{1}$$

Denote*

$$\gamma^+(f) = \lim_{x \to +\infty} \frac{\ln |F(x)|}{e^{-\pi x/2\alpha}}, \qquad \gamma^-(f) = \lim_{x \to -\infty} \frac{\ln |F(x)|}{e^{\pi x/2\alpha}}.$$

Definition 2. $I_\gamma^+ (-\infty < \gamma \leqslant 0)$ is the totality of all $f(x) \in L(-\infty, \infty; \alpha)$ for which $\gamma^+(f) \leqslant \gamma$; $I_\gamma^- (-\infty < \gamma \leqslant 0)$ is the totality of all $f(x) \in L(-\infty, \infty; \alpha)$ for which $\gamma^-(f) \leqslant \gamma$. Obviously, $I_0^+ = I_0^- = L(-\infty, \infty; \alpha)$; $I_{\gamma_1}^+ \subset I_{\gamma_2}^+$, $I_{\gamma_1}^- \subset I_{\gamma_2}^-$ $(-\infty < \gamma_1 < \gamma_2 \leqslant 0)$; $I_{-\infty}^+ = \bigcap_{\gamma \leqslant 0} I_\gamma^+ = 0$, $I_{-\infty}^- = \bigcap_{\gamma \leqslant 0} I_\gamma^- = 0$.

Theorem 1. I_γ^+ and I_γ^- $(-\infty < \gamma < 0)$ are closed prime ideals of the ring L_e. I_γ^+ consists of those and only those functions $f(x) \in L(-\infty, \infty; \alpha)$ which satisfy the equation

$$\int_{-\infty}^{\infty} f(x-y) \frac{e^{-i\mu_1 x}\, dx}{\Gamma\left(\frac{1}{2} + \frac{2\alpha x i}{\pi}\right)} = 0 \qquad (-\infty < y < \infty), \tag{2}$$

where $\mu_1 = -\dfrac{2\alpha}{\pi} \ln |\gamma|$. I_γ^- consists of those and only those functions $f(x) \in L(-\infty, \infty; \alpha)$ which satisfy the equation

$$\int_{-\infty}^{\infty} f(x-y) \frac{e^{-i\mu_2 x}\, dx}{\Gamma\left(\frac{1}{2} - \frac{2\alpha x i}{\pi}\right)} = 0 \qquad (-\infty < y < \infty), \tag{3}$$

where $\mu_2 = \dfrac{2\alpha}{\pi} \ln|\gamma|$. Every prime ideal contained in $L(-\infty, \infty; \alpha)$ has the form $I^+_{\gamma_1} \cap I^-_{\gamma_2}$, where γ_1, γ_2 do not vanish simultaneously.

Let us note that, for the first time, such continuous systems of prime ideals in rings of analytic functions were indicated by G. E. Shilov (^{5}).

Definition 3. Let $\mathfrak{M}$ be an arbitrary set of functions from $L(-\infty, \infty; \alpha)$. We shall call the **spectrum** (^{6}) of $\mathfrak{M}$, and denote it by $\sigma(\mathfrak{M})$, the set of points of the strip $|\operatorname{Im} z| \leqslant \alpha$ at which the Fourier transforms $F(z)$ of all functions $f(x) \in \mathfrak{M}$ vanish.

Obviously, Theorem 1 gives a description of all ideals $I \subset L(-\infty, \infty; \alpha)$ with empty spectrum.

Theorem 2. Let $I \subset L(-\infty, \infty; \alpha)$ be a closed ideal whose spectrum consists of a finite number of interior points of the strip $|\operatorname{Im} z| < \alpha$: $\sigma(I) = \{z_\nu\}$, $|z_\nu| < \alpha$ $(\nu = 1, 2, \dots, k)$. Let n_ν $(\nu = 1, 2, \dots, k)$ be the largest natural number n having the property

$$F(z_\nu) = F'(z_\nu) = \dots = F^{(n-1)}(z_\nu) = 0,$$

where $F(z)$ is the Fourier transform of any function $f(x) \in I$. Put $\gamma_1 = \sup \gamma^+(f)$, $\gamma_2 = \sup \gamma^-(f)$ $(f \in I)$. Then I consists of those and only of those functions which belong to $I^+_{\gamma_1} \cap I^-_{\gamma_2}$ and whose Fourier transforms have at z_ν zeros of multiplicity not less than n_ν $(\nu = 1, 2, \dots, k)$.

Theorems 1 and 2 give a negative answer to a number of questions posed by Mackey (^{7}), p. 105; (^{8}), p. 395).

2°. Each closed ideal I of the ring $L(-\infty, \infty; \alpha)$ determines a certain subspace $J \subset M(-\infty, \infty; \alpha)$, consisting of functions $g(x)$ for which

$$\int_{-\infty}^{\infty} g(x) f(x)\, dx = 0 \quad (f \in I).$$

It is easy to see that J is a weakly closed* subspace of $M(-\infty, \infty; \alpha)$ invariant with respect to the shift operation. We shall call such subspaces ideals of the space $M(-\infty, \infty; \alpha)$. From the general theory of Banach spaces it follows that the correspondence $I \to J$ is a one-to-one mapping of the set of all ideals of $L(-\infty, \infty; \alpha)$ (including 0 and the whole space) onto the set of all ideals of $M(-\infty, \infty; \alpha)$. We shall denote by J^+_γ, J^-_γ $(-\infty \leq \gamma \leq 0)$ the ideals of the

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* It is not difficult to show that the numbers $\gamma^+(f)$ and $\gamma^-(f)$ are always finite and nonpositive $(f \neq 0)$.

space $M(-\infty, \infty; \alpha)$ corresponding to the ideals I_γ^+, I_γ^- of the ring $L(-\infty, \infty; \alpha)$. Clearly,

$$J_{-\infty}^+ = J_{-\infty}^- = M(-\infty, \infty; \alpha), \qquad J_0^+ = J_0^- = 0.$$

Theorem 3. The ideal J_γ^+ $(-\infty < \gamma < 0)$ consists of functions of the form

$$e^{-i\mu_1 x}\chi^+(x) / \Gamma\left(\frac{1}{2} + \frac{2\alpha x i}{\pi}\right) \qquad \left(\mu_1 = -\frac{2\alpha}{\pi}\ln|\gamma|\right),$$

where $\chi^+(x)$ is the boundary value of a function $\chi^+(z)$, holomorphic and bounded in the half-plane $\operatorname{Im} z < 0$. The ideal J_γ^- $(-\infty < \gamma < 0)$ consists of functions of the form

$$e^{-i\mu_2 x}\chi^-(x) / \Gamma\left(\frac{1}{2} - \frac{2\alpha x i}{\pi}\right) \qquad \left(\mu_2 = \frac{2\alpha}{\pi}\ln|\gamma|\right),$$

where $\chi^-(x)$ is the boundary value of a function $\chi^-(z)$, holomorphic and bounded in the half-plane $\operatorname{Im} z > 0$. Any pair of ideals $J_{\gamma_1}^+$, $J_{\gamma_2}^-$ $(-\infty < \gamma_1, \gamma_2 \leq 0)$ has only one common element—the zero of the space $M(-\infty, \infty; \alpha)$.

In particular, J_γ^+ contains all functions

$$e^{-i\mu x} / \Gamma\left(\frac{1}{2} + \frac{2\alpha x i}{\pi}\right)$$

$(\mu_1 \leq \mu < \infty)$, and the ideal J_γ^- contains all functions

$$e^{-i\mu x} / \Gamma\left(\frac{1}{2} - \frac{2\alpha x i}{\pi}\right)$$

$(-\infty < \mu \leq \mu_2)$.

It can be shown that J_γ^+ and J_γ^- are the weakly closed linear hulls of the corresponding sets of these functions.

From the point of view of the spectral theory of rapidly increasing functions (¹), Theorem 3 solves one of the simplest problems of harmonic synthesis. A more general result is given by Theorem 4.

Theorem 4. The general form of functions $g(x) \in M(-\infty, \infty; \alpha)$ whose harmonic spectrum consists of a finite set $\{z_\nu\}_1^k$ of interior points of the strip $|\operatorname{Im} z| \leq \alpha$, and whose nonharmonic spectra are $[\mu_1, \infty)$ and $(-\infty, \mu_2]$, is given by the formula

$$g(x) = \sum_{\nu=1}^{k} P_\nu(x) e^{-iz_\nu x} + \tilde{g}(x), \tag{4}$$

where $P_\nu(x)$ are polynomials; $\tilde{g}(x) \in J^+_{\gamma_1} + J^-_{\gamma_2}$** ($\gamma_1 = -e^{-\pi\mu_1/2\alpha}$, $\gamma_2 = -e^{\pi\mu_2/2\alpha}$).

* In the sense of the topology determined by the weak convergence of the functionals $g(f)$.

** Thus we denote the linear weakly closed hull of the set $J^+_{\gamma_1} \cup J^-_{\gamma_2}$, i.e. the minimal weakly closed subspace of $M(-\infty, \infty; \alpha)$ containing $J^+_{\gamma_1} \cup J^-_{\gamma_2}$.

If one or both of the nonharmonic spectra of $g(x)$ are empty, one should put, respectively, $\mu_1 = +\infty$, or $\mu_2 = -\infty$, or both. The representation (4) is unique.

Let us note that this theorem is "dual" to Theorem 2 and is easily obtained from it.

From Theorem 4 it follows immediately:

Theorem 5. Let the integral equation be given

$$\int_{-\infty}^{\infty} f(x-y)g(y)\,dy = 0, \tag{5}$$

where the kernel $f(x) \in L(-\infty, \infty; \alpha)$, and the unknown function $g(x) \in M(-\infty, \infty; \alpha)$.

If the Fourier transform

$$F(z) = \int_{-\infty}^{\infty} f(x)e^{-izx}\,dx \quad (|\operatorname{Im} z| \leqslant \alpha)$$

has only a finite number of zeros $\{z_\nu\}_1^k$, with $|z_\nu| < \alpha$ ($\nu = 1, 2, \dots, k$), and n_ν are the multiplicities of these zeros, then the general solution of equation (5) is given by the formula

$$g(x) = \sum_{\nu=1}^{k} P_\nu(x)e^{iz_\nu x} + \tilde{g}(x),$$

where $P_\nu(x)$ is a polynomial of degree $\leqslant n_\nu - 1$, and $\tilde{g}(x) \in J^+_{\gamma_1} + J^-_{\gamma_2}$ ($\gamma_1 = \gamma^-(f)$, $\gamma_2 = \gamma^+(f)$).

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Kiev
Civil Engineering Institute

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REFERENCES CITED

[1] B. I. Korenblyum, DAN, **111**, No. 2 (1956).
[2] I. M. Gelfand, D. A. Raikov, G. E. Shilov, Uspekhi Mat. Nauk, **1**, No. 2 (12), 48 (1946).
[3] N. Wiener, Ann. of Math., **33**, 1 (1932).
[4] G. E. Shilov, *On regular normed rings*, Moscow-Leningrad, 1947.
[5] G. E. Shilov, Uchen. Zap. MGU, **146**, 126 (1950).
[6] J. E. Segal, Trans. Am. Math. Soc., **61**, 69 (1947).
[7] D. V. Mackey, Uspekhi Mat. Nauk, **8**, No. 4 (56), 95 (1953).
[8] G. W. Mackey, Bull. Am. Math. Soc., **56**, 385 (1950).

Note: Figure translations are in progress. See original paper for figures.