Several Teaching Points Regarding the General-Solution Formula for First-Order Linear Ordinary Differential Equations

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The general solution formula for first-order linear differential equations is a very important point of knowledge. This paper presents several aspects of this formula that need to be emphasized in the teaching process. The author believes that our work will help readers gain a better understanding of this formula.

Full Text

Several Teaching Points Regarding the General-Solution Formula for First-Order Linear Ordinary Differential Equations

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Abstract The general-solution formula for a first-order linear differential equation is a very important point of knowledge. This paper presents several aspects of this formula that need to be emphasized in the teaching process. The author believes that our work will help readers gain a better understanding of this formula.

[**Keywords**] first-order linear differential equation; general-solution formula; teaching

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1 Introduction

A first-order linear ordinary differential equation refers to an equation of the following form:

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{1}$$

The general-solution formula for equation (1) can be found in most textbooks

1–2

:

$$y = e^{-\int P(x)\,dx} \left(\int Q(x) e^{\int P(x)\,dx}\,dx + C \right). \tag{2}$$

It is generally obtained by the following two methods: (1) the method of variation of constants

1

; (2) the integrating-factor method

2

. Owing to space limitations, the content in this regard can be found in the relevant textbooks and will not be discussed further in this paper.

Using formula (2) to obtain directly the general solution of equation (1) is highly important in practical applications. For example, the general solution of the well-known Bernoulli equation can be obtained with the aid of the general solution of a certain first-order linear ordinary differential equation

1–2

. Some special solutions of the Bernoulli equation play an important role in finding exact solutions of certain soliton equations

3

. Therefore, learning and mastering formula (2) is very important. However, for learners encountering formula (2) for the first time, two questions must first be answered in the course of computation:

Question 1 Why is the integral sign in formula (2) written as an indefinite integral?

Question 2 What should be noted when using formula (2) to solve equation (1)?

The second part of the paper discusses these two questions.

2 Answers to the Questions

Question 1 can equivalently be understood as: why is the integral sign in formula (2) not written as an indefinite integral sign? Here the author offers his own explanation. First, all the indefinite integrals in the formula are in fact definite integrals. This can be seen from the derivation of the formula. On the other hand, formally, the general solution of a first-order linear ordinary differential equation contains only one arbitrary constant, and the general-solution formula has already given one arbitrary constant C; therefore it cannot be an indefinite integral. However, if we use definite integ-

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If an integral sign is used, then the upper and lower limits of the definite integral must be explicitly specified in the formula. Taking the lower limit of the definite integral as an example, can the lower limit be written uniformly as some constant? Here the author gives a simple counterexample.

Example 1

$$\frac{dy}{dx} - \frac{1}{x-a}y = x - a.$$

It is not difficult to obtain its general solution as $y = (x-a)(x+C)$. However, when using the formula to compute $-\int P(x)\,dx = \int \frac{dx}{x-a}$, we find that the point $x = a$ cannot be included in its interval of integration. This is because a is a singular point of this definite integral, and the definite integral diverges at this point. Since a is arbitrary, the general-solution formula has no uniformly valid lower limit of integration. Similarly, one can also conclude from this that the upper limit of integration cannot be determined uniformly.

Problem 2 refers to several matters requiring attention in the process of applying the formula. Here the author needs to point out two of them.

First point The $e^{-\int P(x)\,dx}$ and $e^{\int P(x)\,dx}$ in the formula must be reciprocals of each other when computed.

The author takes the following first-order linear ordinary differential equation as an example:

Example 2

$$\frac{dy}{dx} + 2xy = e^{-x^2}.$$

Its general solution is $y = e^{-x^2}(x+C)$. Suppose that, when computing the first definite integral, we take the first integration constant to be A, namely $e^{-\int P(x)\,dx} = e^{-\int 2x\,dx} = e^{-x^2+A}$. Then, when computing the other definite integral, its value must satisfy the requirement of being the reciprocal, namely $e^{\int P(x)\,dx} = e^{\int 2x\,dx} = e^{x^2-A}$. To illustrate the issue, suppose instead that the integration constant is taken to be B, namely $e^{\int P(x)\,dx} = e^{\int 2x\,dx} = e^{x^2+B}$. Substituting into the general-solution formula gives

$$
\begin{aligned}
y &= e^{-\int P(x)\,dx}\left(\int Q(x)e^{\int P(x)\,dx}\,dx + C\right) \\
&= e^{-x^2+A}\left(\int e^{-x^2}e^{x^2+B}\,dx + C\right) \\
&= e^{-x^2+A}\left(\int e^{B}\,dx + C\right) = e^{-x^2+A}\left(e^{B}x + C\right) = e^{-x^2+A+B}\left(x + Ce^{-B}\right).
\end{aligned}
$$

Substituting this solution into the original equation for verification shows that the function obtained by the above calculation is a general solution of the equation if and only if $A + B = 0$.

The condition $A + B = 0$ is precisely what was mentioned above. Therefore, to avoid such errors, in higher mathematics teaching the author generally advises students either to take all constants of integration directly as 0, or not to compute the other integral, but instead to take the reciprocal of the result of the first integral as its value.

Second point When computing $e^{-\int P(x)\,dx}$, the choice of its value is allowed to differ by a plus or minus sign.

This tells us that, when computing $e^{-\int P(x)\,dx}$, the sign of the function obtained may be arbitrary. We first explain the reason below. In fact, if we let $e^{-\int P(x)\,dx} = \varphi(x)$, then we can obtain the general solution

$$
y = \varphi(x)\left(\int \frac{Q(x)}{\varphi(x)}\,dx + C\right),
$$

then, replacing $\varphi(x)$ in the general solution formula at this point by $-\varphi(x)$, we obtain

$$
y = -\varphi(x)\left(\int \frac{Q(x)}{-\varphi(x)}\,dx + C\right) = \varphi(x)\left(\int \frac{Q(x)}{\varphi(x)}\,dx - C\right).
$$

By the arbitrariness of the integration constant C, the two general solutions are in fact identical.

The second point can sometimes be used to simplify the solution process for some first-order linear ordinary differential equations, as shown in the following example.

Example 3

$$
\frac{dy}{dx} - \cot x \cdot y = 2x\sin x.
$$

According to **the first point**, directly taking the arbitrary constant of the indefinite integral to be 0 gives

$$e^{-\int P(x)\,dx} = e^{\int \cot x\,dx} = e^{\ln|\sin x|} = |\sin x|.$$

However, when calculating the integral inside the parentheses in the general solution formula, we would need to consider cases for this absolute value, which is rather troublesome. But by **the second point**, we know that we do not need to expand and discuss this absolute value; instead, without loss of generality, we may directly take

$$e^{-\int P(x)\,dx} = \sin x$$

and substitute it into the general solution formula to obtain

$$y = e^{-\int P(x)\,dx}\left(\int Q(x)e^{\int P(x)\,dx}\,dx + C\right) = \sin x \cdot \left(\int \frac{2x\sin x}{\sin x}\,dx + C\right)$$

$$= \sin x \cdot \left(\int 2x\,dx + C\right) = \sin x \cdot \left(x^2 + C\right).$$

3 Conclusion

This paper presents several points requiring attention when studying and using the general solution formula for first-order linear ordinary differential equations. As far as the literature currently consulted by the author is concerned, these points have not been explicitly indicated. The author believes that our work is meaningful for beginners and is also helpful for relevant teachers when teaching this knowledge point.

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Teaching Notes on the General Solution Formula to Linear

Ordinary Differential Equation of First Order

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Abstract: The general solution formula for a first-order linear ordinary differential equation is a very important knowledge point. This paper presents several notes that need to be emphasized during the teaching process. We believe that our work will help people gain a better understanding of this formula.

Key words: first-order linear ordinary differential equation; general solution formula; teaching