A positive answer to a symmetry conjecture on homogeneous IFS

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Abstract

We positively answer 'Open Question 1' in the paper of Feng and Wang [Adv. Math. 222 (2009)].

Full Text

Preamble

A POSITIVE ANSWER TO A SYMMETRY CONJECTURE ON HOMOGENEOUS IFS JUNDA ZHANG Abstract. We positively answer 'Open Question 1' in the paper of Feng and Wang [Adv. Math. 222 (2009)].

Our theorem is a positive answer to [1, Open Question 1] as follows.

Theorem 0.1. Let Φ and Ψ be two homogeneous IFSs satisfying the OSC, with the same attractor K R, and the common contraction factor being opposite to each other. Then K is symmetric.

This question was partially answered in [1] under the COSC, and in [2] under the SSC. Our strategy is different. Although our proof seem to be short, we have tried several different strategies using tools from many areas of math to tackle this problem. It turns out that other strategies are much more involved, lengthy but unsatisfying, which does not give the desired result.

We use two lemmas. We first state an easy but useful one. i=1 and B = {bi}n Lemma 0.2. Let A = {ai}n i=1 be two multisets of real numbers satisfying that A + rA = B − rB, where r > 0, A + rA denotes the multiset {ai + ra j : 1 ≤ i, j ≤ n} and B − rB denotes the multiset {bi − rb j : 1 ≤ i, j ≤ n}. Suppose further that bi − ai equals to the same number C for all i. Then A = C(1 − r) Proof. Define the generating functions A(x) = n(cid:88) B(x) = n(cid:88) Since bi = ai + C for all i, we have B(x) = xCA(x).

The multiset equality A + rA = B − rB implies the equality of their generating functions:

A(x)A(xr) = B(x)B(x−r). Compute the right-hand side:

B(x)B(x−r) = (cid:0)xCA(x)(cid:1)(cid:0)x−rCA(x−r)(cid:1) = xC(1−r)A(x)A(x−r).

Thus, A(x)A(xr) = xC(1−r)A(x)A(x−r). Since A(x) is nonzero, we have Now set t = xr, so that x = t1/r and A(xr) = xC(1−r)A(x−r).

A(t) = tC(1−r)/rA(t−1). Date: March 10, 2026.

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It follows that ZHANG n(cid:88) tai = tC(1−r)/r n(cid:88) t−ai = n(cid:88) tC(1−r)/r−ai.

This equality of Laurent polynomials means that the multisets of exponents on both sides coincide.

Hence, {ai : i = 1, . . . , n} = − ai : i = 1, . . . , n (cid:40) C(1 − r) (cid:41) The second one is the key observation.

Lemma 0.3. Let A = {ai}n i=1 be two sets of real numbers with their elements arranged in increasing order: a1 < a2 < · · · < an and b1 < b2 < · · · < bn. Assume that the set A + rA = B − rB has cardinality n2, i.e., all their elements are distinct, where r > 0. Further assume that both sets rB + A and −rA + B have cardinality n2. Then for every 1 ≤ i ≤ n, i=1 and B = {bi}n Proof. Since A + rA = B − rB, we may write bi − rbn = ai + ra1. bk − rbn = as(k) + rat(k), ak + ra1 = bu(k) − rbv(k).

We claim that s and u are permutations from 1 to n.

Rewrite the equation bk − rbn = as(k) + rat(k) as Suppose that there exist k (cid:44) k′ such that s(k) = s(k′). Then bk − rat(k) = as(k) + rbn. as(k) + rbn = as(k′) + rbn, so we have bk − rat(k) = bk′ − rat(k′).

Since the set −rA + B has n2 distinct elements, the mapping (i, j) (cid:55)→ −rai + b j is injective, it follows that t(k) = t(k′) and k = k′, a contradiction. Therefore s is injective, thus a permutation. The fact that u is a bijection follows similarly. Rewrite the equation ak + ra1 = bu(k) − rbv(k) as Suppose that there exist k (cid:44) k′ such that u(k) = u(k′), then bu(k) − ra1 = ak + rbv(k). ak + rbv(k) = bu(k) − ra1 = bu(k′) − ra1 = ak′ + rbv(k′).

Since the set A + rB has n2 distinct elements, it follows that (k, v(k)) = (k′, v(k′)), i.e., k = k′, a contradiction.

Now we prove the lemma by induction on i. Simultaneously, we prove that s(i) = i, t(i) = 1, u(i) = i, and v(i) = n for all i. i = 1. Since the smallest element of A + rA = {ai + ra j : 1 ≤ i, j ≤ n} is a1 + ra1 and Base case: the smallest element of $B - rB = \{bi - rb j : 1 \leq i, j \leq n\}$ is $b1 - rbn$ (because $bi \geq b1$, $b j \leq bn$, and $r > 0$), by the condition $A + rA = B - rB$, we have $b1 - rbn = a1 + ra1$.

Thus, the base case holds. Inductive step: Assume that for all $j < i$, we have b j $-$ rbn = a j + ra1, We prove the statements for step i. s(j) = j, t(j) = 1, u(j) = j, v(j) = n.

A POSITIVE ANSWER TO A SYMMETRY CONJECTURE ON HOMOGENEOUS IFS First, we show that $bi - rbn = ai + ra1$.

Indeed, recall $ai + ra1 = bu(i) - rbv(i)$. Since u is a bijection and u(j) = j for j $<$ i by the inductive hypothesis, we have $u(i) \geq i$. Hence, $bu(i) \geq bi$, so $ai + ra1 = bu(i) - rbv(i) \geq bi - rbv(i) \geq bi - rbn$, where in the last inequality we use the fact that bn is the largest element of B. Similarly, recall $bi - rbn = as(i) + rat(i)$. Since s is a bijection and s(j) = j for $j < i$ by the inductive hypothesis, we have $s(i) \geq i$. Hence, $as(i) \geq ai$, so $bi - rbn = as(i) + rat(i) \geq ai + rat(i) \geq ai + ra1$, showing the desired.

Next, we determine s(i), t(i), u(i), and v(i). It follows that $ai + ra1 = bi - rbn = as(i) + rat(i)$.

By our assumption that $A + rA$ is a set of cardinality n2, we have $s(i) = i$ and $t(i) = 1$. Similarly, $bi - rbn = bu(i) - rbv(i)$, thus $u(i) = i$ and $v(i) = n$.

This completes the inductive step. By induction, the statement holds for all $i = 1, \ldots, n$.

Proof of the Theorem. We write $\Phi = rx + A$ and $\Psi = -rx + B$, where $r > 0$. By Moran' s equation, we can write $A = \{ai\}n$ i=1 with their elements arranged in increasing order, and we know that i=1 and $B = \{bi\}n$ $\Phi \circ \Psi = \Psi \circ \Phi$ satisfies the OSC by [1, Proposition 2.2], thus the set $rB + A = -rA + B$ has cardinality n2, i.e., all their elements are distinct. By [1, Proposition 2.1], we know that satisfies the OSC, thus the set $\Phi \circ \Phi = \Psi \circ \Psi$ $A + rA = B - rB$ has cardinality n2. Then by the second lemma, for every $1 \leq i \leq n$, Then by the first lemma, taking $C = rbn + ra1$, we know that $bi - rbn = ai + ra1$. showing that K is symmetric (see for example, the last line in [1, Proof of Lemma 3.4], which states that the attractor of a homogeneous IFS $\rho x + A$ is symmetric when A is symmetric).

$A = C(1 - r)$

References

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