Linear stability and generalized resonance analysis of crossflow in the boundary layer over a swept flat plate (postprint)

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Abstract

Crossflow instability dominates boundary-layer turbulence transition near the leading edge of swept wings; therefore, its stability analysis is crucial for achieving drag reduction, noise reduction, and thermal protection of flight vehicles via laminar-flow control. In this study, a finite-Reynolds-number-based linear stability analysis is employed to investigate the instability characteristics of the Falkner–Skan–Cooke boundary layer and the swept flat-plate boundary layer. By solving the Orr–Sommerfeld equation and the Rayleigh equation for boundary layers with different pressure gradients, the respective roles of viscous and inviscid effects in crossflow instability are elucidated. Furthermore, based on the generalized resonance mechanism involving triad resonance and Bragg scattering, specific wavenumbers are identified for wall roughness elements and steady disturbance modes that can efficiently trigger crossflow boundary-layer instabilities.

Full Text

Linear Stability and Generalized Resonance Analysis of Crossflow in Swept-Plate Boundary Layers

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Abstract

Crossflow instability dominates the laminar-turbulent transition of boundary layers near the leading edge of swept wing surfaces. Therefore, its stability analysis is crucial for achieving drag reduction, noise reduction, and thermal protection through laminar flow control. This study employs linear stability analysis based on finite Reynolds number theory to investigate unstable eigenmodes in Falkner-Skan-Cooke swept-flat-plate boundary layers. By solving the Orr-Sommerfeld and Rayleigh equations for boundary layers under different pressure gradients, the roles of viscous and inviscid effects in crossflow instability are revealed. Based on the generalized resonance mechanism of three-wave resonance and Bragg scattering, this article provides specific wavenumbers for wall roughness elements and steady disturbance modes that can efficiently trigger instability in crossflow boundary layers. The stability analysis results provide a useful reference for transition prediction in crossflow boundary layers.

Keywords: crossflow boundary layer; hydrodynamic stability; generalized resonance; linear stability theory; turbulent transition

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1. Basic Flow Equations

This study employs dimensionless equations for stability analysis of crossflow boundary layers. The Falkner-Skan-Cooke (FSC) boundary layer represents an idealized boundary layer under ideal conditions and serves as an important reference for investigating more complex boundary layers in practical scenarios. The freestream velocity U^* is selected as the characteristic velocity, the boundary layer displacement thickness L^* as the characteristic length, and the density ρ^* as the characteristic density. Variables with superscript asterisks denote dimensional quantities, while those without asterisks are dimensionless. Here, x represents the chordwise direction of the swept plate, y the wall-normal direction, and z the spanwise direction, with U, V, and W denoting the velocity components in these three directions, respectively.

The boundary layer displacement thickness Reynolds number is defined as $Re = \frac{U^* L^*}{\nu^*}$, where ν^* is the kinematic viscosity at infinity. Assuming the boundary layer flow consists of a steady basic flow $(U(x, y), V(x, y), W(x, y), P(x, y))$ and small disturbances $(u'(x, y, z, t), v'(x, y, z, t), w'(x, y, z, t), p'(x, y, z, t))$, we introduce similarity variables $\eta = Re^{1/2} \frac{y}{x}$ into the classical Prandtl boundary layer equations and express the basic flow velocity components as $F(\eta)$ for the chordwise velocity similarity solution and $G(\eta)$ for the spanwise velocity similarity solution.

The control equations for the boundary layer in this study are the nonlinear boundary layer equations:

$$F_{\eta\eta\eta} + \frac{m+1}{2} F F_{\eta\eta} + m(1 - F_\eta^2) = x(F_\eta F_{x\eta} - F_x F_{\eta\eta})$$

$$G_{\eta\eta} + \frac{m+1}{2} F G_\eta = F_\eta G_x - F_x G_\eta$$

with boundary conditions $F_\eta \to 1$, $G \to \tan\Lambda$ as $\eta \to \infty$, and $F = F_\eta = G = 0$

at $\eta = 0$, where x_0 is the inlet location on the plate, m is the dimensionless acceleration coefficient, and Λ is the sweep angle.

2. Linear Stability Theory

According to classical linear stability theory, crossflow disturbances are assumed to be normal modes:

$$\phi'(x,y,z,t) = \hat{\phi}(y)\exp\left[i\left(\int \alpha(x)dx + \beta z - \omega t\right)\right]$$

where $\alpha = \alpha_r + i\alpha_i$ is the chordwise wavenumber (with real part α_r and imaginary part α_i corresponding to chordwise growth rate), β is the spanwise wavenumber, and ω is the disturbance frequency.

Introducing the parallel flow assumption and substituting into the linearized Navier-Stokes equations yields the control equations for linear stability analysis:

$$\begin{aligned}
& i(\alpha\hat{u} + \beta\hat{w}) + \frac{d\hat{v}}{dy} = 0 \\
& i(\alpha U + \beta W - \omega)\hat{u} + \hat{v}\frac{dU}{dy} = -i\alpha\hat{p} + \frac{1}{Re}\left(\frac{d^2}{dy^2} - \alpha^2 - \beta^2\right)\hat{u} \\
& i(\alpha U + \beta W - \omega)\hat{v} = -\frac{d\hat{p}}{dy} + \frac{1}{Re}\left(\frac{d^2}{dy^2} - \alpha^2 - \beta^2\right)\hat{v} \\
& i(\alpha U + \beta W - \omega)\hat{w} + \hat{v}\frac{dW}{dy} = -i\beta\hat{p} + \frac{1}{Re}\left(\frac{d^2}{dy^2} - \alpha^2 - \beta^2\right)\hat{w}
\end{aligned}$$

This homogeneous system is equivalent to the Orr-Sommerfeld and Squire equations and satisfies homogeneous boundary conditions. For inviscid instability problems, viscous terms are omitted from the control equations to obtain the inviscid linear stability analysis equations (Rayleigh equations).

3. Numerical Methods and Verification

For both the basic flow equations and disturbance equations, discretization in the wall-normal y direction employs compact finite difference schemes. The nonlinear boundary layer equations are solved using the Newton-Raphson method with iterative marching in the x direction, while the eigenvalue problem is solved using the Muller method.

To verify the reliability of the numerical method, we first reproduce the test case from Schrader et al. [15] for a swept-plate boundary layer with inlet parameters $m = 0.34207$, $\Lambda = 45^\circ$, and $Re = 337.9$. Using the nonlinear boundary layer equations, we obtain the basic flow distribution along x and the variation of local boundary layer displacement thickness Reynolds number Re_δ along x. The computed inviscid streamline direction angle ϕ and other results agree well with the numerical results of Schrader et al. [15].

Subsequently, using crossflow modal parameters $\omega = -0.00466$, $\beta = -0.102$, and $\alpha = 0.34207$ as initial values for the eigenvalue problem, we compute the corresponding crossflow modal eigenvalues and eigenfunctions. The results show excellent agreement with those calculated by Schrader et al. [15]. These test cases fully validate the reliability of the numerical methods for basic flow and linear stability theory computations used in this study.

[Figure 2: see original paper] shows the numerical methods verification results.

4. Stability Analysis

Classical crossflow instability theory suggests that the flow is inviscidly unstable. However, in the near-leading-edge region, the Orr-Sommerfeld equation yields growth rates much smaller than those from the Rayleigh equation, indicating that viscosity plays a stabilizing role for crossflow modes.

To investigate the roles of viscous and inviscid effects in crossflow instability, we perform stability analysis on swept-plate boundary layers with the same inlet parameters ($m = 0.34207$, $\Lambda = 45°$) using both Orr-Sommerfeld and Rayleigh equations. [Figure 3: see original paper] presents the evolution of unstable crossflow modal growth rates $-\alpha_i$ in the x direction.

The results show that for fixed frequency $\omega = -0.00466$, the smaller the absolute value of spanwise wavenumber β, the smaller the crossflow modal growth rate. In the downstream region where x is large, the solutions from the Orr-Sommerfeld and Rayleigh equations gradually converge, consistent with classical crossflow instability theory. For larger favorable pressure gradients (e.g., $m = 0.5$), the crossflow modal growth rates are greater than those for $m = 0.34207$ under the same frequency and wavenumber conditions, because $m = 0.5$ corresponds to a stronger favorable pressure gradient.

[Figure 4: see original paper] compares the neutral curves for swept-plate boundary layers and FSC boundary layers with the same inlet parameters. The neutral curves for both cases are similar, with unstable crossflow modal regions appearing near the leading edge. For steady disturbances ($\omega = 0$), the unstable spanwise wavenumber region gradually decreases and shifts toward smaller absolute β values as the disturbance propagates chordwise. The unstable region in the FSC boundary layer is larger than that in the swept-plate boundary layer because the FSC boundary layer has a constant favorable pressure gradient, whereas the swept-plate boundary layer' s favorable pressure gradient is concentrated primarily in the leading-edge region.

[Figure 5: see original paper] shows the N-factor values for unstable crossflow modes calculated using the eN method. The maximum N-factor appears downstream of $x = 700$ and moves further downstream as frequency decreases. This suggests that lower frequencies have larger unstable regions, causing the neutral curve boundary to shift downstream. The N-factor distribution also shows selectivity in spanwise wavenumber, with the maximum growth occurring at

$\beta = -0.35$ for lower frequencies, while being less sensitive to frequency.

5. Generalized Resonance

In crossflow boundary layers, unstable disturbances and wall roughness elements can satisfy three-wave resonance conditions and effectively excite and amplify unstable waves [18]. Three-wave resonance includes both difference resonance and sum resonance. For difference resonance, the frequencies and wavenumbers must satisfy:

$$\omega_1 = \omega_2 - \omega_3, \quad \alpha_1 = \alpha_2 - \alpha_3, \quad \beta_1 = \beta_2 - \beta_3$$

For sum resonance, they must satisfy:

$$\omega_1 = \omega_2 + \omega_3, \quad \alpha_1 = \alpha_2 + \alpha_3, \quad \beta_1 = \beta_2 + \beta_3$$

[Figure 6: see original paper] and Table 1 show combinations of unsteady crossflow modes and wall roughness elements satisfying three-wave resonance relations in swept-plate and FSC boundary layers. Unlike three-wave resonance in two-dimensional boundary layers that requires quasi-neutral conditions [19], crossflow boundary layers can support generalized resonance between unstable disturbances and wall roughness elements [18]. Theoretical analyses [20] have revealed that sum resonance is a more effective instability mechanism than difference resonance. The strongest resonance amplification occurs for steady modes or roughness elements with wavenumbers near the right-branch neutral steady mode.

When the participating crossflow unstable mode is steady ($\omega = 0$), the sum resonance condition degenerates into the Bragg scattering relation:

$$\alpha_1 = 2\alpha_2, \quad \beta_1 = 2\beta_2$$

[Figure 7: see original paper] shows the growth rate distribution and spanwise wavenumber distribution along x for steady neutral modes participating in Bragg scattering. The steady neutral modes with larger absolute spanwise wavenumbers correspond to the strongest sum resonance. The absolute value of spanwise wavenumber gradually decreases along the chordwise direction. Steady crossflow modes or wall roughness elements with spanwise wavenumbers near the most unstable steady mode produce the strongest Bragg scattering resonance.

6. Conclusions

Due to the favorable pressure gradient existing primarily in the leading-edge region of swept plates, strong instability occurs in this region. This study employs Orr-Sommerfeld and Rayleigh equations based on finite Reynolds number theory to analyze crossflow stability in swept-plate boundary layers. The analysis reveals that the leading-edge region is dominated by inviscid instability, with the unstable spanwise wavenumber region gradually decreasing and shifting toward smaller absolute β values along the chordwise direction. Near the leading-edge stagnation point, the unstable frequency region gradually increases along the chordwise direction.

Using the eN method based on linear stability theory, we calculate the N-factor distribution for unstable crossflow disturbances in the downstream region. The crossflow modal growth in swept-plate boundary layers is less sensitive to frequency but shows sensitivity to spanwise wavenumber, with maximum growth occurring at $\beta = -0.35$. In contrast, FSC boundary layers are sensitive to both frequency and spanwise wavenumber.

Based on the generalized resonance mechanism, we compute the spanwise wavenumber distributions along the chordwise direction for the upper-branch neutral steady mode that can excite the strongest sum resonance, and for the most unstable steady mode that can excite Bragg scattering. These results demonstrate that steady modes or wall roughness elements near the most unstable steady modal wavenumbers can excite faster unstable growth.

References

[1] SHEN Luyu, CHANGGEN Lu. Receptivity of steady cross-flow vortices in three-dimensional boundary layer[J]. Acta Physica Sinica, 2017, 66(1): 220-234 (in Chinese).

[2] SHEN Luyu, CHANGGEN Lu, XIAOQING Chen. Numerical study of unstable waves induced by interaction between free-stream turbulence and three-dimensional localized wall roughness in flat-plate boundary layer[J]. Applied Mathematics and Mechanics, 2017, 38(11): 1208-1221.

[3] SHEN Luyu, CHANGGEN Lu. Excitation of linear instability in Falkner-Skan-Cooke boundary layer[J]. Journal of Applied Fluid Mechanics, 2021, 14(5): 1437-1445.

[4] SHEN Luyu, CHANGGEN Lu. Boundary-layer receptivity under interaction of free-stream turbulence and localized roughness[J]. Applied Mathematics and Mechanics, 2016, 37(3): 349-360.

[5] SHEN Luyu, CHANGGEN Lu. Local receptivity of non-parallel boundary layer[J]. Applied Mathematics and Mechanics, 2016, 37(7): 467-480.

[6] XU Jiakuan, SONG Fuquan. Görtler vortices and streaks in boundary layer subject to pressure gradient: excitation by streamwise vortical disturbances, nonlinear evolution and secondary instability[J]. Journal of Fluid Mechanics, 2020, 900: A15.

[7] XU Jiakuan, ZHANG Yan. Nonlinear evolution and secondary instability

of steady and unsteady Görtler vortices induced by free-stream vortical disturbances[J]. Journal of Fluid Mechanics, 2017, 829: 681-730.

[8] BIPPES H. Basic experiments on transition in three-dimensional boundary layers dominated by crossflow instability[J]. Progress in Aerospace Sciences, 1999, 35(4): 363-412.

[9] SARIC W S, WHITE E B. Stability and transition of three-dimensional boundary layers[J]. Annual Review of Fluid Mechanics, 2003, 35: 413-440.

[10] GREGORY N, STUART J T, WALKER W S. On the stability of three-dimensional boundary layers with application to the rotating disk[J]. Philosophical Transactions of the Royal Society of London, Series A: Mathematical and Physical Sciences, 1955, 248(943): 155-199.

[11] CHOUDHARI M. Roughness-induced generation of crossflow vortices in three-dimensional boundary layers[J]. Theoretical and Computational Fluid Dynamics, 1994, 6(1): 1-30.

[12] CHOUDHARI M. Long-wavelength asymptotics of unstable crossflow modes, including effect of surface curvature[J]. Proceedings of the Royal Society of London, Series A: Mathematical and Physical Sciences, 1995, 451(1943): 515-541.

[13] BUTLER T L, SHEN Luyu, CHANGGEN Lu. Stationary crossflow vortices in leading-edge three-dimensional boundary layers: the non-parallelism and excitation of surface roughness[J]. Journal of Fluid Mechanics, 2018, 845: 93-140.

[14] SCHRADER L U, BRANDT L, HENNINGSON D S. Receptivity mechanisms in three-dimensional boundary-layer flows[J]. Journal of Fluid Mechanics, 2009, 618: 209-241.

[15] HE X, BUTLER T L. Effects of distributed roughness on crossflow instability through generalized resonance mechanisms[J]. Journal of Fluid Mechanics, 2019, 858: 787-831.

[16] CRAIK A D D. Non-linear resonant instability in boundary layers[J]. Journal of Fluid Mechanics, 1971, 50(2): 393-413.

[17] XU Jiakuan. Surface-roughness effects on crossflow instability in swept-wing boundary layers through generalized resonances[J]. AIAA Journal, 2022, 60(5): 2887-2904.

[18] REIBERT M S, SARIC W S, CARRILLO R B. Experiments on nonlinear saturation of stationary crossflow vortices in a swept-wing boundary layer[C]//34th Aerospace Sciences Meeting and Exhibit. Reston, USA: AIAA, 1996: AIAA 1996-0183.

[19] XU Jiakuan. Fully local amplification factor transport equation for stationary crossflow instabilities[J]. AIAA Journal, 2019, 57(7): 2682-2693.

[20] XU Jiakuan, ZHANG Yan. Spatial-temporal transformation of primary and secondary instabilities in weakly non-parallel shear flows[J]. Journal of Fluid Mechanics, 2023, 959: A21.

[21] MACK L M. Boundary-layer linear stability theory[EB/OL]. (1984-06-01)[2024-08-19]. https://ntrs.nasa.gov/citations/19840020174.

[22] GOLDSTEIN M E, DURBIN P A. The effect of finite turbulence spatial scale on the amplification of turbulence in a contracting stream[J]. Journal of Fluid Mechanics, 1980, 98(3): 473-508.

Note: Figure translations are in progress. See original paper for figures.