

Post-buckling imprint of beams on an elastic foundation with finite thickness

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Abstract

Flexible substrate-supported plates are common in nature and engineering, such as skin and asphalt pavements, and buckling is one of their typical failure modes. The lateral support provided by the foundation helps enhance the stability of the plate. When the foundation thickness is comparable to or larger than the buckling wavelength, existing elastic models cannot accurately capture the thickness effect of the foundation, and analytical solutions for the critical buckling strain have not yet been reported. To address this issue, this study investigates thin plates supported by an elastic substrate and, based on linear elasticity theory, proposes an elastic foundation model with finite thickness to examine the buckling of an infinitely long beam resting on such a foundation. Two types of contact between the foundation and a rigid base are considered, namely bonded and sliding. Concise explicit relationships are derived between the critical buckling strain and critical buckling wavelength and the Young's modulus, Poisson's ratio, and thickness of the foundation. The influence of the foundation thickness on the critical buckling wavelength and critical buckling strain is analyzed. Finite element analyses are carried out to validate the model. The results show that, compared with existing analytical models, the predictions of the present model agree better with the finite element results; when the foundation thickness exceeds $(1/)$ times the buckling wavelength, the relative error is less than 10%.

Full Text

Buckling of Beams Supported by Elastic Foundations of Finite Thickness

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Abstract: Flat plates supported by flexible substrates are commonly found in nature and engineering, such as in skin and asphalt roads, with buckling being one of their prevalent failure modes. The lateral support provided by the foundation enhances plate stability. However, when the foundation thickness is substantial relative to the buckling wavelength, existing elastic models fail to accurately capture the thickness effect of the foundation, and analytical solutions for critical buckling strain remain unavailable. To address this, we propose a finite-thickness elastic foundation model for substrate-supported thin plates based on linear elasticity theory and investigate the buckling of infinitely long beams on such foundations. Two scenarios are examined: foundations bonded to a rigid substrate and foundations sliding on a rigid substrate. We derive concise explicit relationships among critical buckling strain, critical buckling wavelength, and the foundation's Young's modulus, Poisson's ratio, and thickness, and analyze how foundation thickness influences these critical parameters. Finite element analysis validates the model, demonstrating that its predictions agree better with numerical results than existing analytical models. When the foundation thickness exceeds the buckling wavelength, the relative error falls below...

Keywords: elastic foundation; finite thickness; buckling; buckling strain; finite element

1 Analytical Model

Elastic foundation models describe the relationship between surface forces and deformations in foundations. The Winkler model, characterized by linear springs, has evolved into two-parameter models [2-3], three-parameter models [4-5], and half-space elastic models [1]. Due to their simplicity, these elastic foundation models have found widespread application in various elasticity problems. Previous studies have investigated plate buckling on elastic foundations [14-15] and beam buckling on thin elastic layers [18-19]. However, the former is unsuitable for relatively thick foundations, while the latter only provides implicit relationships between critical buckling strain and foundation thickness.

Many materials in biological and engineering fields, such as muscle tissue, exhibit compressive strengths much lower than their Young's modulus, making instability a primary failure mechanism for these structures. Consequently, buckling of beams bonded to elastic foundations has become an active research area [8-12], with investigations covering buckling stability of beams on soil [13], buckling of functionally graded beams on elastic foundations [16], wrinkling and buckling delamination of elastic films on substrates [17], thermal post-buckling of sandwich beams on elastic foundations [18], and thermal buckling of plates on foundations [19]. The Winkler model has also been applied to beam buckling on half-space elastic foundations [20-21]. The lateral forces from elastic foundations enhance beam stability, with the supporting effect depending on the foundation's

s mechanical properties and thickness. While linear elastic models provide accurate descriptions, they present computational challenges. Simplified elastic foundation models are thus essential for precise buckling analysis.

Existing models have limitations: the Winkler model neglects Poisson's ratio effects and cannot handle thick foundations; half-space models cannot describe thickness effects; and multi-parameter models face parameter determination difficulties. This study proposes a finite-thickness elastic foundation model that accurately reveals foundation support effects and yields explicit relationships for critical buckling strain with foundation Young's modulus, Poisson's ratio, and thickness, providing a theoretical basis for analyzing buckling of beams on elastic foundations.

Consider an infinitely wide thin plate perfectly bonded to an elastic foundation of thickness h . The beam and foundation interface experiences no separation or sliding. As the study focuses on macroscopic engineering applications, surface energy is negligible. The plate ends are subjected to axial compressive force N per unit width induced by thermal or mechanical loads. The infinitely wide thin plate can be treated as a slender Euler-Bernoulli beam, while the underlying elastic foundation follows linear elastic plane strain constitutive relations. The x - and y -axis origins are set at the elastic foundation's lower surface, making the foundation's upper surface initially at $y = 0$. The beam's bending stiffness is $D = \frac{E_b t^3}{12(1-\nu_b^2)}$, where E_b is the beam's Young's modulus, ν_b is the beam's Poisson's ratio, t is the beam thickness, E is the foundation's Young's modulus, and ν is the foundation's Poisson's ratio. The foundation's lower surface is either bonded to or slides on a rigid substrate.

Assuming the post-buckling beam shape follows a sinusoidal form $u_y = V_0 \sin(mx)$, where V_0 is a real constant defining the buckling amplitude and m is a positive real constant related to the buckling wavelength λ (with $m = 2\pi/\lambda$). Only vertical displacement of the beam is considered, while axial displacement is neglected per beam theory assumptions. Figure 1 shows the planar geometric model of a beam bonded to an elastic foundation.

[Figure 1: see original paper] Planar geometric model of beam bonded to elastic foundation

1.1 Establishment of Elastic Foundation Model

Based on the defined beam buckling shape, the stress function in the foundation region is assumed as $\Phi = f(y) \sin(mx)$, where $f(y)$ is a real-valued function of y . For a foundation bonded to a rigid substrate, with boundary conditions $u_x = 0$, $u_y = 0$ at the lower surface $y = -h$, and considering $hm \gg 1$, the stress-displacement relationship at the foundation upper surface is derived (see Appendix). For a foundation sliding on a rigid substrate, the boundary conditions become $u_x = 0$, $\tau_{xy} = 0$ at $y = -h$, yielding the stress-strain relationship.

1.2 Beam Control Differential Equation

Assuming constant beam axial force N , the differential relationship between buckling force N and lateral normal stress σ_y provided by the foundation, based on force and moment equilibrium, is [6,22]:

$$\sigma_y = -D \frac{d^4 u_y}{dx^4} - N \frac{d^2 u_y}{dx^2}$$

This control differential equation applies to Euler-Bernoulli beams. By substituting the foundation stress-displacement relationship and the beam buckling shape, the model can be extended to Timoshenko beam stability analysis. However, due to its complexity and since high beams have strong stability and are less prone to instability, this study focuses on infinitely long beam buckling where the Euler-Bernoulli assumption meets accuracy requirements and yields more concise results.

1.3 Critical Buckling Force

To obtain the analytical form of buckling force N , we substitute the stress-displacement relationship into the control equation, yielding:

$$N = Dm^2 + \frac{E}{2(1+\nu)} \cdot \frac{3 + 4hm + 3h^2m^2}{1 + hm}$$

Taking the extremum of N with respect to m gives the critical m value:

$$m_c = \sqrt[4]{\frac{E}{2D(1+\nu)}}$$

The critical buckling wavelength is:

$$\lambda_c = \frac{2\pi}{m_c} = 2\pi \sqrt[4]{\frac{2D(1+\nu)}{E}}$$

And the critical buckling force is:

$$N_c = 2\sqrt{D \cdot \frac{E}{2(1+\nu)}} = \sqrt{\frac{2DE}{1+\nu}}$$

For the sliding foundation case, the buckling force expression becomes:

$$N = Dm^2 + \frac{E}{2(1+\nu)} \cdot \frac{1 + 2hm}{1 + hm}$$

The corresponding critical m is:

$$m_c = \sqrt[4]{\frac{E}{4D(1+\nu)}}$$

The critical buckling wavelength is:

$$\lambda_c = 2\pi \sqrt[4]{\frac{4D(1+\nu)}{E}}$$

And the critical buckling force is:

$$N_c = \sqrt{\frac{DE}{1+\nu}}$$

As foundation thickness approaches infinity ($h \rightarrow \infty$), the leading terms for both cases become identical, indicating that bottom boundary conditions do not affect the critical buckling force at first order, which is physically reasonable. Higher-order terms differ between the two boundary conditions, with the bonded case showing larger critical buckling stress at finite thickness, also consistent with physical expectations.

2 Results and Analysis

Finite element analysis (FEA) is conducted to verify the derived formulas, comparing the proposed model with FEA results and existing half-space elastic foundation models [20]. Notably, even as foundation thickness approaches infinity, the models are not identical because this model assumes zero beam axial displacement while half-space models assume zero shear stress at the foundation surface. Subsequent comparisons show the current model agrees better with FEA results.

2.1 Finite Element Model

A two-dimensional solid model of the foundation-beam system is established in ABAQUS, with perfect bonding between foundation and beam. For comparison with the analytical model based on linear elasticity, both beam and foundation are modeled as homogeneous linear elastic materials with Poisson's ratio 0.3. The Young's modulus ratio between foundation and beam is initially set to 0.01. A linear perturbation buckling analysis step is employed.

For the bonded foundation case, axial displacement is fixed at both ends of the beam and foundation, and a temperature load is applied to induce buckling. The numerical critical buckling strain is obtained from the first mode eigenvalue n as $\epsilon_{cr} = n\Delta T\alpha$, where ΔT is the applied temperature increase and α is

the thermal expansion coefficient. For the sliding foundation case, uniform compressive displacement S/L is applied at the right end while fixing the left end's axial displacement, where L is the beam length. The model considers two common buckling causes in engineering: thermal loading and mechanical compression.

The FEA model uses quadrilateral four-node linear plane strain elements with mesh size $h/20$. The relative error between simulation results from models with mesh sizes $h/20$ and $h/40$ is less than 1.5%, ensuring mesh independence.

2.2 Comparison of Results

Figures 2-7 show variations of dimensionless critical m values and critical buckling strain with foundation Young's modulus and thickness. For both bonded and sliding foundations, the m values and their variation patterns from this model show reasonable agreement with FEA results (relative error $< 5\%$). Compared with existing models, this model accounts for foundation thickness effects.

[Figure 2: see original paper] Variation of critical m value with foundation Young's modulus

The dimensionless critical m value increases with Young's modulus E , causing the critical buckling wavelength to decrease with increasing foundation stiffness.

[Figure 3: see original paper] Variation of critical m value with foundation thickness

[Figure 4: see original paper] Variation of critical buckling strain with foundation Young's modulus ($h/h_b = 10$)

[Figure 5: see original paper] Variation of critical buckling strain with foundation thickness ($E/E_b = 0.01$)

[Figure 6: see original paper] Variation of critical buckling strain with hm ($h/h_b = 10$)

[Figure 7: see original paper] Variation of critical buckling strain with hm ($E/E_b = 0.01$)

Critical buckling strain increases with foundation Young's modulus E , with the bonded foundation showing slightly higher critical buckling strain than the sliding case. The relative error between this model's predictions and numerical solutions is less than 5%. The error increases with foundation Young's modulus because the model neglects beam axial displacement. Compared with existing models, this model's results are closer to numerical solutions.

Critical buckling strain initially decreases rapidly with foundation thickness h and then approaches a constant value when $h \gg h_b$. When $h/h_b \geq 12$, thickness effects become negligible. However, due to simplifications, the analytical solution shows some error compared to FEA results. When $h/h_b < 12$, thickness

effects are significant, and existing models produce larger errors because they neglect these effects in solving for m .

When foundation thickness is much greater than beam thickness, the critical buckling strains for bonded and sliding foundations converge, consistent with the analysis considering thickness effects. In contrast, half-space elastic foundation models [20] do not distinguish between bonded and sliding conditions.

3 Conclusions

This study proposes a finite-thickness elastic foundation model for buckling of beams on elastic foundations, deriving explicit relationships between critical buckling strain and foundation thickness. The model focuses on the simplification using the assumption $hm \gg 1$.

Key findings: 1. When foundation thickness is less than the buckling wavelength, thickness effects are significant. Critical buckling strain decreases with increasing foundation thickness. 2. When foundation thickness exceeds the buckling wavelength, thickness effects become negligible. The model form matches existing half-space elastic foundation models, but predictions remain closer to FEA results. 3. For foundation thickness greater than π times the buckling wavelength, relative errors between model predictions and numerical solutions remain below $2/\pi$.

The model accurately reveals the supporting effect of elastic foundations and provides a theoretical basis for analyzing instability in composite fibers and other applications.

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Appendix: Derivation of Stress-Strain Relationship at Bonded Foundation Surface

According to elasticity theory, the stress function in the foundation region must satisfy the biharmonic equation:

$$\nabla^4 \Phi = 0 \quad (A1)$$

Assuming the form $\Phi = f(y) \sin(mx)$ and substituting into (A1) yields the general solution:

$$f(y) = (A + Cy)e^{my} + (B + Dy)e^{-my} \quad (A2)$$

The stress components are:

$$\sigma_y = m^2 \sin(mx)[(A + Cy)e^{my} + (B + Dy)e^{-my}] \quad (A3)$$

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} = m^2 \sin(mx)[(A + Cy + 2C/m)e^{my} + (B + Dy - 2D/m)e^{-my}]$$

$$\tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} = -m \cos(mx)[(A + Cy + C/m)e^{my} - (B + Dy - D/m)e^{-my}]$$

Under plane strain conditions, the displacement components are:

$$u_x = \frac{1 + \nu}{E} \cos(mx)[(A - B) + (C - D)y] \quad (A4)$$

$$u_y = \frac{1 + \nu}{E} \sin(mx)[-(A + B) - (C + D)y + \frac{2(1 - \nu)}{m}(C - D)] \quad (A5)$$

At the foundation upper surface ($y = 0$), the boundary conditions are $\sigma_y = -p_0 \sin(mx)$ and $u_y = -\frac{p_0}{E} \sin(mx)$. At the bonded lower surface ($y = -h$), $u_x = u_y = 0$. Solving these yields:

$$A - B = \frac{p_0}{2m^2}, \quad C - D = -\frac{p_0}{2m}$$

$$Ch + 2Dhe^{-mh} = 0, \quad (A + Ch) - (B - Dh) = 0$$

Considering $hm \gg 1$ and using asymptotic approximations:

$$C \approx -\frac{p_0}{2m}, \quad D \approx \frac{p_0}{2m} e^{-2mh}$$

$$A - B \approx \frac{p_0}{2m^2}, \quad C + D \approx -\frac{p_0}{2m}$$

Substituting into the stress-displacement relationship at $y = 0$ yields:

$$\sigma_y = -\frac{E(1 + hm)}{2(1 + \nu)(3 + 4hm + 3h^2m^2)} u_y$$

For the sliding foundation case with boundary conditions $u_x = 0$, $\tau_{xy} = 0$ at $y = -h$, similar derivation gives:

$$\sigma_y = -\frac{E(1 + hm)}{2(1 + \nu)(1 + 2hm)} u_y$$

These relationships form the basis for the critical buckling analysis presented in Section 1.

Note: Figure translations are in progress. See original paper for figures.

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