Postprint of "Polynomial Series Solution Method for Nonlinear Bending of Thin Plates Based on Deep Learning Techniques"

Authors: Peng Linxin

Date: 2026-01-30T23:25:31+00:00

Abstract

Based on the theory of elasticity and in combination with deep learning techniques, a polynomial solution method is developed for the geometrically nonlinear bending analysis of thin plates. To satisfy boundary conditions and enable efficient differentiation, a displacement trial function model is first constructed using symbolic differentiation; the input to this model is the coordinates of an arbitrary point within the solution domain, and the output is the corresponding displacement. Then, considering the geometrically nonlinear bending theory derived from the variational principle, the system potential energy of the thin plate is used to formulate the objective function. Finally, automatic differentiation is employed to compute the gradient of the objective function, and the Adam optimization algorithm is used for parameter optimization until the optimal parameter vector is obtained, thereby yielding an explicit polynomial series solution for the plate displacement. The combination of these two differentiation techniques enhances the usability and accuracy of the program. In this study, geometrically nonlinear bending problems of thin plates with different shapes and boundary conditions are solved; the convergence of the polynomial series solutions at different orders is analyzed, and the results are compared with Abaqus finite element solutions and reference solutions from the literature, demonstrating the effectiveness and applicability of the proposed method.

Full Text

Polynomial Series Solution for Nonlinear Bending of Thin Plates Based on Deep Learning Technology

Abstract

Based on the theory of elastic mechanics and combined with deep learning techniques, this study develops a polynomial series solution method for geometrically nonlinear bending analysis of thin plates. To satisfy boundary conditions and enable rapid differentiation, a trial displacement function model is first constructed based on symbolic differential computation. The model takes coordinates of arbitrary points in the solution domain as input and outputs the corresponding displacements. By considering the geometrically nonlinear bending theory based on variational principles, the objective function is constructed from the system' s potential energy of the thin plate. Finally, automatic differentiation is employed to compute the gradient of the objective function, which is then optimized using optimization algorithms until the optimal parameter vector is obtained, yielding an explicit polynomial series solution for plate displacement. The combination of these two differential techniques enhances both the usability and accuracy of the program. This research solves geometrically nonlinear bending problems for thin plates of various shapes and boundary conditions, analyzes the convergence of polynomial series solutions under different orders, and validates the effectiveness and applicability of the proposed method through comparisons with finite element solutions and reference solutions.

Keywords: geometric nonlinearity; symbolic differentiation; automatic differentiation; polynomial series; deep learning

1. Introduction

Thin plates serve as critical engineering components, and their nonlinear bending behavior is more representative of practical engineering scenarios than linear bending. The nonlinear bending problem of plates has long attracted extensive attention from scholars, leading to the development of numerous numerical solution methods. Jiang Tao et al. [?] applied the symplectic superposition method to obtain analytical solutions for orthotropic rectangular plates under uniform loads. Shen Guozhe et al. [?] established a coupled peridynamics and continuum mechanics model for analyzing large deformation bending and fracture of thin plates. Mo Huang [?] investigated the nonlinear bending behavior of functionally graded plates using the spline finite point method. Watts et al. [?] employed a meshless Galerkin method to study geometrically nonlinear bending of quadrilateral composite plates. Chen Yingjie et al. [?] applied a modified reciprocal theorem for rectangular plates to derive generalized displacement solutions for large deflection bending. Zhang [?] obtained approximate solutions for nonlinear bending of super-elliptical plates using the Ritz method. Li [?] solved nonlinear axisymmetric bending problems of strain gradient circular plates using differential quadrature and iterative methods. Yu [?] utilized a hierarchical wavelet method for analyzing geometrically nonlinear anisotropic plates. Al-Shugaa et al. [?] employed perturbation methods to handle boundary and initial conditions for annular plates, deriving analytical expressions for nonlinear large deflection. Chen et al. [?] investigated arbitrary quadrilateral plates under large deflection. Zhang Xunwei et al. [?] analyzed circular plate nonlinear deflection using Gröbner bases. Aghelinejad et al. [?] performed post-buckling analysis of functionally graded annular plates. Jiang Furu [?] studied unsymmetrical nonlinear bending of annular and circular plates under various supports using perturbation methods. Zhou Ciqing [?] derived consistent asymptotic solutions for orthotropic rectangular plates. Yang Tianxiang et al. [?] proposed internal and mixed collocation methods. Chen Ziqi et al. [?] used double trigonometric series as deflection trial functions with least squares collocation. Yang Jiaming et al. [?] assumed displacement functions as beam vibration functions. Wu Lanhe et al. [?] employed products of binary polynomials and basic functions describing boundary shapes. Hou Xianglin et al. [?] applied finite difference methods and dynamic design variable optimization algorithms.

In addition to these conventional methods, with the rapid advancement of computer technology in machine learning, neural network methods have been widely applied to solve partial differential equation-described mechanics problems due to their powerful function approximation capabilities. Raissi et al. [?] proposed Physics-Informed Neural Networks (PINN) for solving nonlinear PDEs. Sirignano et al. [?] developed the Deep Galerkin Method (DGM). Samaniego et al. [?] introduced the Deep Energy Method for PDE solutions in weak form. Goswami et al. [?] combined transfer learning with PINN for phase-field fracture modeling. Yang et al. [?] employed Bayesian PINNs for problems with noisy data. Wang et al. [?] proposed subdomain neural networks for heterogeneous complex geometries. Aghelinejad et al. [?] used deep learning for rectangular plate mechanics. Huang Zhongmin et al. [?] developed a deflection-bending moment coupled neural network for plates with in-plane stiffness gradient. Zhuang et al. [?] proposed a learning-based numerical algorithm for solving variational problems. Nguyen-Thanh et al. [?] applied the Deep Energy Method to hyperelasticity. Guo Hongwei et al. [?] employed deep collocation and energy methods with two-step optimizers for plate bending. Zhang Chun et al. [?] proposed a deep neural network surrogate model for mechanical field quantities.

Neural network models essentially establish functional relationships between inputs and outputs, determining these relationships by continuously optimizing internal parameters through training. The maturity of automatic differentiation techniques has facilitated the development of various methods, enabling easy gradient computation for error functions without designing low-level derivative interfaces. However, the function expressions obtained from trained deep neural networks with numerous parameters and activation functions become extremely complex, making their internal mechanisms difficult to interpret. Additionally, large networks lead to many optimization parameters, causing computational resource waste.

For most problems in elasticity mechanics, many common solution methods exist. Combining these with deep learning techniques can enrich their flexibility. This study draws from traditional elasticity solutions, assuming displacement trial functions in polynomial series form to construct input-output relationships, using thin plate system potential energy as the objective function, and optimizing with common learning algorithms.

Figure 1

Figure 1: Figure 1

2. Thin Plate Geometrically Nonlinear Bending Theory

For nonlinear bending problems of thin plates, the system' s potential energy is derived based on variational principles. Assuming the plate' s elastic modulus E, Poisson' s ratio μ, and thickness t, the deformation potential energy includes bending deformation energy U_{bending} and midplane strain energy U_{midplane}. The plate stiffness is D, and the uniform load is q.

The bending strain energy is expressed in terms of displacement derivatives. The midplane strain components are:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

The total strain energy U combines bending and midplane contributions:

$$U = U_{\text{bending}} + U_{\text{midplane}}$$

The external work W is:

$$W = \int_\Omega qw\, d\Omega$$

Thus, the system' s potential energy Π is:

$$\Pi = U - W$$

Automatic differentiation technology efficiently computes partial derivative terms of displacement in the objective function. By combining common optimization algorithms to optimize parameters, displacement trial functions can be assumed as polynomial series forms for typical boundary conditions:

$$w(x, y) = \sum_{i=1}^{n} A_i \phi_i(x, y)$$

where A_i represents the tensor of parameters to be optimized, and $\phi_i(x, y)$ are distance functions satisfying boundary conditions.

3. Polynomial Series Model for Displacement Trial Functions

The coefficient tensor A_i contains n terms of the polynomial series. The polynomial series model adopted in this study is shown in

. Based on different boundary conditions, displacement trial functions for the plate are constructed. Symbolic differentiation builds derivative function models for the trial functions. When data points are input into the model, predictions for displacement w and its derivatives $w_x, w_{xx}, ...$ are obtained. Automatic differentiation technology then creates the objective function (the plate' s system potential energy) and obtains the relationship between system potential energy and parameter tensors for optimization algorithms to find optimal parameter vectors.

For large deflection bending problems, displacement trial functions are assumed as polynomial series forms. Symbolic differentiation rapidly solves explicit expressions for displacement trial functions and their derivatives, which are stored as code segments for subsequent calls. To enable the model to learn the load-displacement relationship, gradient descent algorithms iteratively optimize model parameters. This study divides the load into steps, performing the above iteration process at each load step. Except for the first iteration step, model parameters for subsequent steps come from the previous iteration.

The series model is built through symbolic differential operations by the program. By changing parameter n, the computational framework of the model can be constructed.

4. Loss Function

The loss function consists solely of the plate' s system potential energy:

$$L = \Pi$$

Boundary conditions are handled by distance functions $\phi_i(x, y)$. For simple essential boundary conditions, special solutions are constructed. For complex boundary conditions, the loss function must incorporate boundary terms. Essential boundary conditions include simply supported and clamped boundaries:

$$\Gamma_{\text{clamped}} : w = 0, \quad \frac{\partial w}{\partial n} = 0$$

$$\Gamma_{\text{simply supported}} : w = 0, \quad M_n = 0$$

where n represents the outward normal direction of the boundary.

5. Parameter Optimization

The model training process is essentially parameter optimization. Automatic differentiation technology computes the gradient of the error function with respect to parameter tensor A, which is then combined with optimization algorithms for error backpropagation. Through continuous iterative training, the optimal parameter tensor A^* is obtained to minimize the error function:

$$A^* = \arg\min_A L(A)$$

When the parameter vector reaches A^*, the displacement output by the model represents the true displacement of the plate.

6. Optimization Algorithm

The optimization algorithm uses: - Loss function L - Learning rate α - Threshold ϵ

The algorithm computes gradient values $g_t \leftarrow \partial L/\partial A$ using automatic differentiation, updates parameters A, and repeats until convergence criteria are met. The learning rate is adjusted between 10^{-4} and 10^{-2} based on actual computations.

7. Numerical Examples

All examples in this study use the STRI3 element in Abaqus to discretize the solution domain with a sufficient number of elements to obtain converged solutions. Comparison results are expressed by relative error:

$$e = \frac{|u - u^*|}{|u^*|} \times 100\%$$

where u^* represents the Abaqus solution or literature solution, and u represents the solution from this study.

7.1 Triangular Plate A simply supported triangular plate is shown in [FIGURE:2] with side length $a = 1$ m, thickness $t = 0.01$ m, elastic modulus $E = 3$ MPa, Poisson' s ratio $\mu = 0.3$, and uniform load q. Both small deflection and large deflection bending are solved.

The displacement trial functions are constructed as:

$$\phi_i(x, y) = (x + y + 1)(x - y + 1)(2x - y - 2)$$

For small deflection bending, the theoretical solution is [**?**]:

$$w = \frac{q}{64D}(x^2 + y^2 - a^2)(x^2 + y^2 - 2a^2)$$

The coefficients obtained by this method compare well with theoretical solutions (see). Due to symmetry, w is an even function of y, so odd-power coefficients of y in the theoretical solution are zero, and the optimized odd coefficients also reach minimal values.

Parameter optimization results for large deflection bending with $q = 80$ N/m^2 are shown in . The load-displacement curve is presented in [FIGURE:3].

7.2 Square Plate A simply supported square plate under uniform load is shown in [FIGURE:4] with elastic modulus E and thickness t. The dimensionless formulas are:

$$q^- = \frac{qa^4}{Et^4}, \quad w^- = \frac{w}{t}$$

Using second-order polynomial series, displacement trial functions are constructed. The optimization results for parameters in each displacement expression are shown in the table. The load-displacement curve matches well with the finite element solution.

To demonstrate accuracy, compares dimensionless deflections at the center point with literature and Abaqus solutions, showing good agreement with relative errors below 2%.

The convergence of polynomial series order is analyzed in [FIGURE:5]. When using polynomial series above second order, the optimized displacement functions converge well to literature and Abaqus solutions, indicating good approximation of the true displacement function.

7.3 Parallelogram Plate A parallelogram plate with angle 60°, side length 0.5 m, thickness $t = 0.01$ m, elastic modulus $E = 3 \times 10^4$ MPa, Poisson' s ratio $\nu = 0.3$, and uniform load $q = 100$ N/m^2 is shown in [FIGURE:6].

Different boundary conditions are denoted as: - CCCC: all edges clamped - SSSS: all edges simply supported - SCSC: two opposite edges simply supported, others clamped - SFSC: one edge free, others simply supported/clamped

Distance functions $\phi_i(x, y)$ are constructed accordingly. Convergence analysis for each boundary condition is shown in [FIGURE:7]. For CCCC, SSSS, and SCSC boundaries, polynomial series above first order converge well. For SFSC with free edges, higher-order polynomials (above fifth order) are required for convergence, as complex stress boundary conditions necessitate more complex polynomial expressions.

compares deflections at the midpoint of the free edge for SFSC boundary, showing decreasing relative errors as polynomial order increases.

8. Discussion and Analysis

The proposed algorithm follows a similar 流程 to the deep energy method, with the main difference being model construction. While neural network models can represent extremely complex functions and theoretically approximate any function form, their internal mechanisms remain difficult to interpret. In contrast, the polynomial series model constructed through symbolic differentiation has clear applicability for large deflection bending problems of thin plates, yielding explicit polynomial series expressions for displacement functions.

Symbolic differentiation for computing partial derivatives of displacement is more efficient than automatic differentiation. However, as polynomial series order increases, computation time and memory consumption increase overall. shows the impact of polynomial order on computing time and memory for different cases.

9. Conclusions

This study proposes a polynomial series method based on deep learning technology for solving geometrically nonlinear bending of thin plates. The main conclusions are:

1. A numerical model combining automatic differentiation and symbolic differentiation is developed. The method yields explicit polynomial series solutions and demonstrates effectiveness and accuracy for both linear and nonlinear bending problems, with solutions matching well with Abaqus.
2. Convergence analysis shows that for most boundary conditions, lower-order polynomial series can approximate the true displacement function. However, for free boundaries, higher-order polynomials are required for convergence.
3. The model framework is simple to implement, reusable, and can be extended to other mechanics problems by increasing polynomial order.

References

[1] Jiang Tao, Eburilitu. Symplectic superposition solution for bending problem of orthotropic rectangular plate with adjacent edges clamped and others free[J]. Chinese Journal of Applied Mechanics, 2020, 37(5): 2214-2221.

[2] Shen Guozhe, Ruiyang Yang, et al. Large deformation and fracture analysis of plate bending based on peridynamics[J]. Chinese Journal of Theoretical and Applied Mechanics, 2023, 55(2): 381-389.

[3] Mo Huang. Spline finite point method for nonlinear bending of functionally graded plates[J]. Advanced Materials Research, 2011, 18(5): 319-326.

[4] Watts G, Pradyumna S, Singh A. Nonlinear analysis of quadrilateral composite plates using moving kriging based element-free Galerkin method[J]. Composite Structures, 2017, 159: 719-735.

[5] Chen Yingjie, Xiaohui Zhang. Problem of bending of large deflection rectangular plate solved by applying corrected reciprocal theorem[J]. Journal of Plasticity Engineering, 2018, 25(2): 175-182.

[6] Zhang L. Non-linear bending analysis of super-elliptical plates[J]. International Journal of Non-Linear Mechanics, 2013, 48: 157-163.

[7] Li Y. Nonlinear axisymmetric bending analysis of strain gradient circular plate[J]. Applied Mathematical Modelling, 2021, 1: 363-380.

[8] Yu T. Hierarchical wavelet method for nonlinear bending of materially geometrically anisotropic plate[J]. Communications in Nonlinear Science and Numerical Simulation, 2021, 92: 105498.

[9] Al-Shugaa M, Al-Gahtani H. Transversely loaded anisotropic composite plates undergoing large deflection[J]. Arabian Journal for Science and

Engineering, 2023, 48(4): 5571-5586.

[10] Chen L, Lanhe W. Bending analysis of arbitrary quadrilateral plates with large deflection[J]. Chinese Quarterly of Mechanics, 2000, 21(3): 322-326.

[11] Cao Tianjie. Derivation and optimization algorithm of high-order nonlinear PDE for plate bending large deformation[J]. Acta Physica Sinica, 2012, 61(18): 9-18.

[12] Zhang Xunwei, Jianfang Y, Jiliang D. Using Gröbner bases to analyze nonlinear deflection of circular plate[J]. Manufacturing Automation, 2013, 35(6): 82-85.

[13] Aghelinejad B, Ebrahimi M. Nonlinear thermomechanical post-buckling analysis of functionally graded annular plates based on von-Karman' s plate theory[J]. Mechanics of Advanced Materials and Structures, 2020, 2020(1): 2943705.

[14] Jiang Furu. Unsymmetrical nonlinear bending problems of annular and circular plates under various supports (I)[J]. Applied Mathematics and Mechanics, 1982, 3(5): 629-640.

[15] Zhou Ciqing. Nonlinear bending of orthotropic rectangular plates[J]. Applied Mathematics and Mechanics, 1984, 5(3): 419-436.

[16] Yang Tianxiang, et al. Solution of large deflection bending problem of rectangular plates by collocation method[J]. Journal of Tianjin University, 1981(2): 7-19.

[17] Chen Ziqi, Cida Y. Analysis of geometrically nonlinear plate bending problems by least squares collocation method[J]. Acta Mechanica Sinica, 1983(5): 522-526.

[18] Yang Jiaming, Liangxin H. A general method for nonlinear bending of orthotropic rectangular plates[J]. Chinese Quarterly of Mechanics, 2002, 23(4): 568-574.

[19] Wu Lanhe, Chunyu L. Bending analysis of arbitrary quadrilateral thin plates with large deflection[J]. Chinese Quarterly of Mechanics, 2000, 21(3): 322-326.

[20] Hou Xianglin, Xijian Z, Liang Z. Study on dynamic design variable optimization algorithm for large deflection of thin plates[J]. Chinese Journal of Applied Mechanics, 2016, 33(3): 427-433.

[21] Raissi M, Perdikaris P, Karniadakis G. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations[J]. Journal of Computational Physics, 2019, 378: 686-707.

[22] Sirignano J, Spiliopoulos K. DGM: A deep learning algorithm for solving partial differential equations[J]. Journal of Computational Physics, 2018, 375: 1339-1364.

[23] Samaniego E, Anitescu C, Goswami S. An energy approach to the solution of partial differential equations in computational mechanics by machine learning: Concepts, implementation and applications[J]. Computer Methods in Applied Mechanics and Engineering, 2020, 362: 112790.

[24] Goswami S, Anitescu C, Chakraborty S. Transfer learning enhanced physics informed neural network for phase-field modeling of fracture[J]. Theoretical and Applied Fracture Mechanics, 2020, 106: 102447.

[25] Yang Y, Karniadakis G. B-PINNs: Bayesian physics-informed neural networks for forward and inverse problems with noisy data[J]. Journal of Computational Physics, 2021, 425: 109913.

[26] Wang Y, Cen S. CENN: Conservative energy method based on neural networks for heterogeneous complex geometries[J]. Computer Methods in Applied Mechanics and Engineering, 2022, 389: 114250.

[27] Aghelinejad B, et al. Deep learning method for solving forward and inverse mechanics problems of rectangular plates[J]. Chinese Journal of Computational Mechanics, 2022, 39(1): 120-128.

[28] Huang Zhongmin, et al. Deflection-bending moment coupling neural network method for bending problem of plates with in-plane stiffness gradient[J]. Chinese Journal of Theoretical and Applied Mechanics, 2021, 53(9): 2541-2553.

[29] Zhuang X, et al. A deep learning-based energy method for bending, vibration and buckling analysis of Kirchhoff plates with transfer learning[J]. European Journal of Mechanics-A/Solids, 2021, 87: 104225.

[30] Zhuang X, Alajlan N. Autoencoder based energy method for solving partial differential equations: Application to heterogeneous materials[J]. Computer Methods in Applied Mechanics and Engineering, 2020, 362: 112790.

[31] Nguyen-Thanh V, Zhuang X, Rabczuk T. A deep learning method for finite deformation hyperelasticity[J]. European Journal of Mechanics-A/Solids, 2020, 80: 103874.

[32] Guo Hongwei, Zhuang X. The application of collocation method and energy method with two-step optimizer for Kirchhoff plate bending analysis[J]. Chinese Journal of Solid Mechanics, 2021, 42(3): 249-266.

[33] Zhang Chun, Dengpeng L. Research on surrogate calculation model of mechanical quantity field based on neural network[J]. Chinese Journal of Applied Mechanics, 2021, 38(2): 552-559.

[34] Wang Xiaomin, et al. Intelligent crack identification based on neural network[J]. Chinese Journal of Applied Mechanics, 2022, 39(6): 1158-1167.

[35] Timoshenko S, Woinowsky-Krieger S. Theory of Plates and Shells[M]. New York: McGraw-Hill College, 1959.